2022 ATE accelerator school

가속기 물리 II

POSTECH
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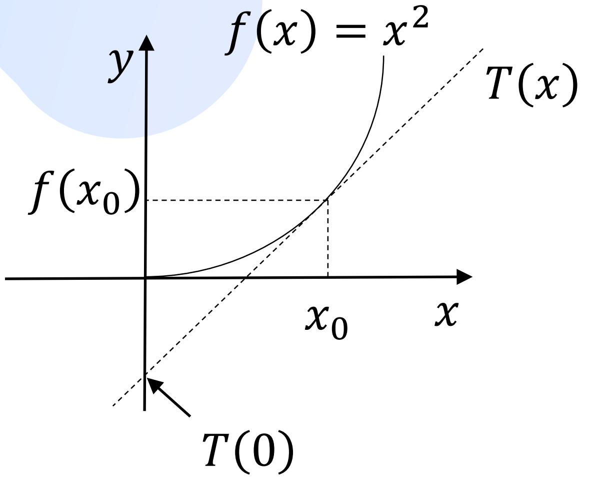
Today’s agenda

- Legendre transform을 이해한다.
- Energy function을 이해한다.
- 해밀토니안의 의미를 이해하고 운동 방정식을 유도한다.

Legendre transformation – 기하학적 접근법

Question) For a given function $f(x) = x^2$, define $f'(x) = u$, Find $g(u)$ such that $\frac{\partial g}{\partial u} = -x$.

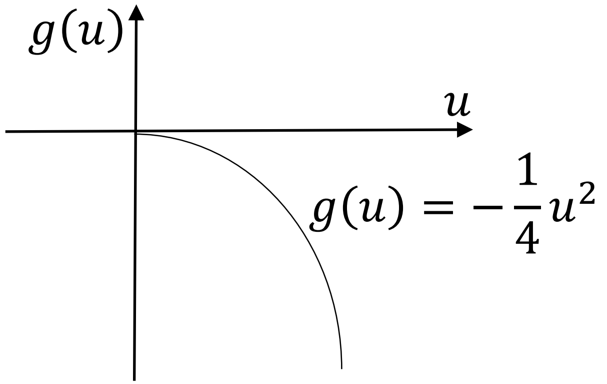


$$T(x) = f'(x_0) \cdot (x - x_0) + f(x_0)$$

$$T(0) = f'(x_0) \cdot (-x_0) + f(x_0) = -x_0 f'(x_0) + f(x_0) \equiv g(x_0)$$

$$g(x_0) = -x_0 f'(x_0) + f(x_0) = -2x_0^2 + x_0^2 = -x_0^2$$

$$f'(x_0) = 2x_0 \equiv u \quad \Rightarrow \quad x_0 = \frac{u}{2}$$



$$g(u) = -\left(\frac{u}{2}\right)^2 = -\frac{u^2}{4}$$

$$\frac{\partial g}{\partial u} = -\frac{u}{2} = -x$$

Legendre transformation – 대수적 접근법 (1-D)

Question) For a given function $f(x) = x^2$ with $f'(x) = u$, Find $g(u)$ such that $\frac{\partial g}{\partial u} = -x$.

$$df = \frac{\partial f}{\partial x} dx = udx = d(ux) - xdu$$

$$d(f - ux) = -xdu \quad \Rightarrow \quad dg = -xdu \quad \Rightarrow \quad g = g(u), \quad \therefore \frac{\partial g}{\partial u} = -x$$

$$f - ux \equiv g$$

$$u = f'(x) = 2x \quad \Rightarrow \quad x = \frac{u}{2}$$

$$g(u) = f(x) - ux = x(u)^2 - ux(u) = \frac{u^2}{4} - u \times \frac{u}{2} = -\frac{u^2}{4}$$

$$\therefore \frac{\partial g}{\partial u} = -\frac{u}{2} = -x$$

Legendre transformation – 대수적 접근법 (2-D)

Question) For a given function $f(x,y)$ with $\frac{\partial f}{\partial x} = u$, Find $g(u,y)$ such that $\frac{\partial g}{\partial u} = -x$.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = udx + vdy = d(ux) - xdu + vdy$$

$$d(f - ux) = -xdu + vdy \Rightarrow dg = -xdu + vdy \Rightarrow g = g(u,y)$$

$$f - ux \equiv g$$

$$\therefore \frac{\partial g}{\partial u} = -x$$

$$\therefore \frac{\partial g}{\partial y} = v = \frac{\partial f}{\partial y}$$

$$g(u,y) = f(x,y) - ux$$

Lagrangian 의 이해 1

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(\mathcal{L}) - \frac{\partial}{\partial q_j}(\mathcal{L}) = 0 \qquad \mathcal{L} = T - V$$

If $\frac{\partial}{\partial q_j}(\mathcal{L}) = 0$ ➡ q_j is cyclic.

then, $\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(\mathcal{L}) = 0$ ➡ $\frac{\partial}{\partial \dot{q}_j}(\mathcal{L}) = \text{constant}$

$\frac{\partial}{\partial \dot{q}_j}(\mathcal{L}) \equiv p_j$: Generalized momentum.

Ex 1) Cartesian coordinate

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad V = f(x, y)$$

$$\frac{\partial}{\partial z}(\mathcal{L}) = \frac{\partial}{\partial z}(T - V) = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{z}}(\mathcal{L}) = \frac{d}{dt}(m\dot{z}) = 0$$

➡ $m\dot{z} \equiv p_z = const.$

Linear momentum conservation

Ex 2) Polar coordinate

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad V = f(r)$$

$$\frac{\partial}{\partial \theta}(\mathcal{L}) = \frac{\partial}{\partial \theta}(T - V) = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}(\mathcal{L}) = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

➡ $mr^2\dot{\theta} \equiv L_z = const.$

Angular momentum conservation

Lagrangian 의 이해 2

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(\mathcal{L}) - \frac{\partial}{\partial q_j}(\mathcal{L}) = 0 \qquad \mathcal{L} = T - V$$

If $\frac{\partial \mathcal{L}}{\partial t} = 0,$

$$\frac{d}{dt}\left(-\mathcal{L} + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\dot{q}_j\right) = 0$$

$$\frac{d\mathcal{L}(q,\dot{q};t)}{dt} = \sum_j \frac{\partial \mathcal{L}}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial \mathcal{L}}{\partial t}$$

$$h = -\mathcal{L} + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\dot{q}_k$$ Energy function

$$\frac{d\mathcal{L}(q,\dot{q};t)}{dt} = \sum_j \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial \mathcal{L}}{\partial t}$$

$$\frac{d\mathcal{L}(q,\dot{q};t)}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\dot{q}_j\right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$\frac{d}{dt}\left(\mathcal{L} - \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\dot{q}_j\right) = \frac{\partial \mathcal{L}}{\partial t}$$

Energy function의 이해

Cartesian 좌표계에서 energy function의 값이 energy와 같다는 것을 상대론이 적용되지 않는 경우와 상대론이 적용되는 경우로 나누어서 보이시오.

$$h = -\mathcal{L} + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q_k}} \dot{q_k} \qquad \mathcal{L}(q, \dot{q}) = T - V$$

Non-relativistic case

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\sum_k \frac{\partial \mathcal{L}}{\partial \dot{q_k}} \dot{q_k} = \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{x} + \frac{\partial \mathcal{L}}{\partial \dot{y}} \dot{y} + \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{z} = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 2T$$

$$h = -\mathcal{L} + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q_k}} \dot{q_k} = -\mathcal{L} + 2T = T + V = E$$

Law of Thermodynamics

Frist law: $dU = -dW + dQ$ → energy conservation

Second law: $\Delta S \geq 0$ for isolated system.

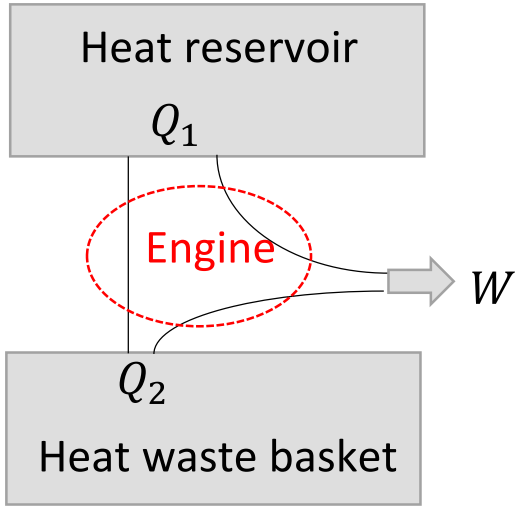
: $dS = \frac{dQ}{T}$ for not isolated system.

U : Internal energy

W: Work done by the system

Q : Heat absorbed by the system

S : Entropy



$$dW = PdV$$

$$W = \int_{V_i}^{V_f} P(V)dV$$

$$dU = -dW + dQ = -PdV + TdS$$

$$= \left(\frac{\partial U}{\partial V}\right)_S dV + \left(\frac{\partial U}{\partial S}\right)_V dS$$

$$U = U(V, S)$$

$$P = -\left(\frac{\partial U}{\partial V}\right)_S \qquad T = \left(\frac{\partial U}{\partial S}\right)_V$$

Helmholtz free energy

P : Pressure

V : Volume

T : Temperature

F : Helmholtz free energy

$$U = U(V, S)$$

$$dU = -PdV + TdS$$

$$= -PdV + d(TS) - SdT$$

$$dU - d(TS) = -PdV - SdT$$

$$d(U - TS) = -PdV - SdT \quad \Rightarrow \quad dF = -PdV - SdT$$

$$F = F(V, T)$$

$$F = U - TS$$

$$dF = \left(\frac{\partial F}{\partial V}\right)_T dV + \left(\frac{\partial F}{\partial T}\right)_V dT \quad \Rightarrow \quad P = -\left(\frac{\partial F}{\partial V}\right)_T \qquad S = -\left(\frac{\partial F}{\partial T}\right)_V$$

Legendre transformation과의 관계

Legendre transformation

$$f = f(x,y)$$

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

$$= udx + vdy$$

$$g = f - ux$$

$$dg = -xdu + vdy$$

$$dg = \left(\frac{\partial g}{\partial u}\right)_y du + \left(\frac{\partial g}{\partial y}\right)_u dy$$

$$u = \left(\frac{\partial f}{\partial x}\right)_y \qquad v = \left(\frac{\partial f}{\partial y}\right)_x$$

$$x = -\left(\frac{\partial g}{\partial u}\right)_y \qquad v = \left(\frac{\partial g}{\partial y}\right)_u$$

$u \leftrightarrow T$

$v \leftrightarrow -P$

$g \leftrightarrow F$

$x \leftrightarrow S$

$y \leftrightarrow V$

Helmholtz free energy

$$U = U(V,S)$$

$$dU = \left(\frac{\partial U}{\partial V}\right)_S dV + \left(\frac{\partial U}{\partial S}\right)_V dS$$

$$= TdS - PdV$$

$$F = U - TS$$

$$dF = -PdV - SdT$$

$$dF = \left(\frac{\partial F}{\partial V}\right)_T dV + \left(\frac{\partial F}{\partial T}\right)_V dT$$

$$T = \left(\frac{\partial U}{\partial S}\right)_V \qquad -P = \left(\frac{\partial U}{\partial V}\right)_S$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \qquad -P = \left(\frac{\partial F}{\partial V}\right)_T$$

P : pressure

V : Volume

T : Temperature

F : Helmholtz free energy

Hamiltonian

Energy function $h(q,\dot{q},t) = -\mathcal{L}(q,\dot{q},t) + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q_k}} \dot{q_k}$ → Similar to Legendre transformation

Thermodynamics	Mechanics
$F(V,T) = U(V,S) - TS$	$h(q,\dot{q},t) = -\mathcal{L}(q,\dot{q},t) + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q_k}} \dot{q_k}$
V	q
S	$\dot{q}$
T	$\frac{\partial \mathcal{L}}{\partial \dot{q_k}} \equiv p_k$
$U(V,S)$	$-\mathcal{L}(q,\dot{q},t)$
$F(V,T)$	$\mathcal{H}(q,p_k,t) = -\mathcal{L}(q,\dot{q},t) + \sum_k p_k \dot{q_k}$

Hamiltonian

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_k} dq_k + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$\frac{\partial \mathcal{L}}{\partial q_k} \equiv v_k \qquad \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \equiv p_k \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

$$= v_k dq_k + p_k d\dot{q}_k + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$= v_k dq_k + d(p_k \dot{q}_k) - \dot{q}_k dp_k + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$d(\mathcal{L} - p_k \dot{q}_k) = v_k dq_k - \dot{q}_k dp_k + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$d(-\mathcal{L} + p_k \dot{q}_k) = -v_k dq_k + \dot{q}_k dp_k - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$-\mathcal{L} + p_k \dot{q}_k = \mathcal{H}$$

$$d\mathcal{H} = -v_k dq_k + \dot{q}_k dp_k - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$= \frac{\partial \mathcal{H}}{\partial q_k} dq_k + \frac{\partial \mathcal{H}}{\partial p_k} dp_k + \frac{\partial \mathcal{H}}{\partial t} dt$$

$$\frac{\partial \mathcal{H}}{\partial q_k} = -v_k = -\frac{\partial \mathcal{L}}{\partial q_k} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = -\frac{dp_k}{dt} = \dot{p}_k$$

$$\frac{\partial \mathcal{H}}{\partial q_k} = \dot{p}_k$$

$$\frac{\partial \mathcal{H}}{\partial p_k} = \dot{q}_k$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

Hamilton's equation of motion

Hamiltonian

$$\mathcal{H}(q_k, p_k, t) = -\mathcal{L}(q_k, \dot{q}_k, t) + \sum_k p_k \dot{q}_k$$

	Lagrangian $\mathcal{L}$	Hamiltonian $\mathcal{H}$
Variables	$q_k, \dot{q}_k, t$	q_k, p_k, t
Function	$\mathcal{L}(q, \dot{q}, t)$	$\mathcal{H}(q, p, t)$
Variable change	$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$	$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k}$
Function change	$\mathcal{L} = -\mathcal{H} + \sum_k \dot{q}_k p_k$	$\mathcal{H} = -\mathcal{L} + \sum_k p_k \dot{q}_k$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial q_k} = -\frac{\partial \mathcal{H}}{\partial q_k}$$

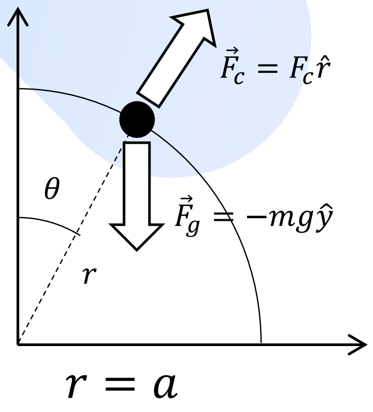
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{dp_k}{dt} = \dot{p}_k$$

$$\dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k}$$

Hamiltonian's equation of motion

Find equation of motion for the particle in the figure by using Hamiltonian when the particle is moving with non-relativistic speed.



$$\mathcal{L}(\theta,\dot{\theta},t) = T - V = \frac{1}{2}ma^2\dot{\theta}^2 - amg\cos(\theta)$$

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\theta} = ma^2\dot{\theta} \quad \rightarrow \quad \dot{\theta} = \frac{p_\theta}{ma^2}$$

$$\mathcal{H} = -\mathcal{L} + p_\theta\dot{\theta} = -\frac{1}{2}ma^2\left(\frac{p_\theta}{ma^2}\right)^2 + amg\cos(\theta) + p_\theta\frac{p_\theta}{ma^2}$$

$$= \frac{1}{2}\frac{{p_\theta}^2}{ma^2} + amg\cos(\theta)$$

$$\dot{q}_k = \frac{\partial\mathcal{H}}{\partial p_k}$$

$$\dot{p}_k = -\frac{\partial\mathcal{H}}{\partial q_k}$$

$$\frac{d\theta}{dt} = \frac{\partial\mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{ma^2} \quad \rightarrow \quad p_\theta = ma^2\dot{\theta} = a \times ma\dot{\theta} = \vec{r} \times \vec{p}_{mech}$$

Angular momentum

$$\frac{dp_\theta}{dt} = -\frac{\partial\mathcal{H}}{\partial\theta} = amg\sin(\theta) \quad \rightarrow \quad \frac{d^2\theta}{dt^2} = \frac{1}{ma^2}amg\sin(\theta) = \frac{g}{a}\sin(\theta)$$

Hamiltonian for relativistic particle

$$\mathcal{L} = -\frac{mc^2}{\gamma} - V = -mc^2\sqrt{1-\frac{(\dot{x}^2+\dot{y}^2+\dot{z}^2)}{c^2}} - V(x,y,z)$$

In Cartesian coordiante

Canonical momentum variable for x axis

$$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = -mc^2\cdot\gamma\cdot\left(-\frac{2\dot{x}}{c^2}\right) = \gamma m\dot{x} \qquad p_y = \frac{\partial\mathcal{L}}{\partial\dot{y}} = \gamma m\dot{y} \qquad p_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = \gamma m\dot{z}$$

$$\mathcal{H} = -\mathcal{L} + \sum_k p_k\,\dot{q}_k = -\mathcal{L} + p_x\dot{x} + p_y\dot{y} + p_z\dot{z} = -\mathcal{L} + \gamma m\dot{x}^2 + \gamma m\dot{y}^2 + \gamma m\dot{z}^2 = -\mathcal{L} + \gamma m v^2$$

$$= \frac{mc^2}{\gamma} + V + \gamma m v^2 = mc^2\sqrt{1-\frac{v^2}{c^2}} + V + \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mc^2\left(1-\frac{v^2}{c^2}\right) + mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + V = \gamma mc^2 + V$$

$$\mathcal{H} = \sqrt{\gamma^2 m^2 v^2 c^2 + m^2c^4} + V = \sqrt{\left(p_x{}^2 + p_y{}^2 + p_z{}^2\right)c^2 + m^2c^4} + V$$

숙제

) $\gamma mc^2 = \sqrt{\gamma^2 m^2 v^2 c^2 + m^2 c^4}$

Hamiltonian for relativistic particle

$$\mathcal{L} = -\frac{mc^2}{\gamma} - V = -mc^2\sqrt{1 - \frac{(\dot{r}^2 + r^2\dot{\theta}^2)}{c^2}} - V$$

In Polar coordinate

Canonical momentum variable for x axis

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = -mc^2 \cdot \gamma \cdot \left(-\frac{2\dot{r}}{c^2}\right) = \gamma m \dot{r}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -mc^2 \cdot \gamma \cdot \left(-\frac{2r^2\dot{\theta}}{c^2}\right) = \gamma m r^2 \dot{\theta}$$

$$\mathcal{H} = -\mathcal{L} + \sum_k p_k \dot{q}_k = -\mathcal{L} + p_r\dot{r} + p_\theta\dot{\theta} = -\mathcal{L} + \gamma m\dot{r}^2 + \gamma m r^2\dot{\theta}^2 = -\mathcal{L} + \gamma m v^2$$

$$= \frac{mc^2}{\gamma} + V + \gamma m v^2 = \gamma m c^2 + V$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \frac{{p_r}^2}{\gamma^2 m^2} + r^2\frac{{p_\theta}^2}{\gamma^2 m^2 r^4}$$

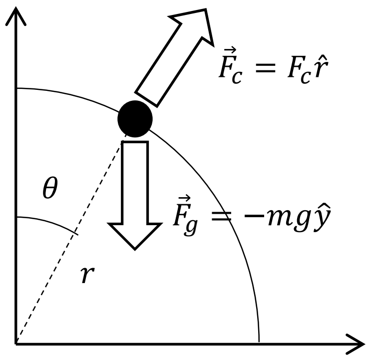
$$= \frac{1}{\gamma^2 m^2}\left({p_r}^2 + \frac{1}{r^2}{p_\theta}^2\right)$$

$$\mathcal{H} = \sqrt{\gamma^2 m^2 v^2 c^2 + m^2 c^4} + V$$

$$= \sqrt{\left({p_r}^2 + \frac{1}{r^2}{p_\theta}^2\right)c^2 + m^2c^4} + V(r, \theta)$$

Example

Find equation of motion for the particle in the figure by using Hamiltonian when the particle is moving with relativistic speed.



$$\mathcal{H} = \gamma mc^2 + V = \sqrt{\left({p_r}^2 + \frac{1}{r^2}{p_\theta}^2\right)c^2 + m^2c^4} + V(r,\theta)$$

$$p_r = \gamma m\dot{a} = 0$$

$$p_\theta = \gamma ma^2\dot{\theta}$$

$$r = a$$

$$V(a,\theta) = mga\cos(\theta)$$

$$\mathcal{H} = \sqrt{\frac{c^2}{a^2}{p_\theta}^2 + m^2c^4} + mga\cos(\theta)$$

$$\frac{d\theta}{dt} = \frac{\partial\mathcal{H}}{\partial p_\theta} = \frac{1}{2}\frac{2\frac{c^2}{a^2}p_\theta}{\sqrt{\left(\frac{1}{a^2}{p_\theta}^2\right)c^2 + m^2c^4}} = \frac{\frac{c^2}{a^2}p_\theta}{\gamma mc^2} \quad \Rightarrow \quad p_\theta = \gamma ma^2\dot{\theta}$$

$$\frac{dp_\theta}{dt} = -\frac{\partial\mathcal{H}}{\partial\theta} = mga\sin(\theta) \quad \Rightarrow \quad \dot{\gamma}ma^2\dot{\theta} + \gamma ma^2\ddot{\theta} = mga\sin(\theta)$$

Hamiltonian for charged particle

$$\mathcal{L} = T - qV + q\vec{v}\cdot\vec{A} = T - qV + q\dot{x}A_x + q\dot{y}A_y + q\dot{z}A_z$$

$$\mathcal{H} = -\mathcal{L} + \sum_k p_k\,\dot{q}_k = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - T + qV - q\dot{x}A_x - q\dot{y}A_y - q\dot{z}A_z$$

$$= (p_x - qA_x)\dot{x} + \left(p_y - qA_y\right)\dot{y} + (p_z - qA_z)\dot{z} - T + qV$$

In Cartesian coordinate

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

1. Non-relativistic case

$$\mathcal{H} = (p_x - qA_x)\dot{x} + \left(p_y - qA_y\right)\dot{y} + (p_z - qA_z)\dot{z} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qV$$

$$= m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qV$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qV = T + qV$$

$$= \frac{1}{2m}\left((p_x - qA_x)^2 + (p_x - qA_x)^2 + (p_x - qA_x)^2\right) + qV$$

$$\equiv \frac{\left(\vec{p} - q\vec{A}\right)^2}{2m} + qV$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Canonical momentum variable for x axis

$$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x} + qA_x$$

$$p_y = \frac{\partial\mathcal{L}}{\partial\dot{y}} = m\dot{y} + qA_y$$

$$p_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z} + qA_z$$

Hamiltonian for charged particle

In Cartesian coordinate

$$\mathcal{L} = T - qV + q\vec{v}\cdot\vec{A} = T - qV + q\dot{x}A_x + q\dot{y}A_y + q\dot{y}A_y$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$\mathcal{H} = (p_x - qA_x)\dot{x} + (p_y - qA_y)\dot{y} + (p_z - qA_z)\dot{z} - T + qV$$

2. Relativistic case

$$\mathcal{H} = (p_x - qA_x)\dot{x} + (p_y - qA_y)\dot{y} + (p_z - qA_z)\dot{z} + \frac{mc^2}{\gamma} + qV$$

$$= \gamma m\dot{x}^2 + \gamma m\dot{y}^2 + \gamma m\dot{z}^2 + \frac{mc^2}{\gamma} + qV$$

$$= \gamma m v^2 + \frac{mc^2}{\gamma} + qV = \gamma mc^2 + qV$$

$$= \sqrt{\gamma^2 m^2 v^2 c^2 + m^2c^4} + qV$$

$$= \sqrt{\gamma^2 m^2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)c^2 + m^2c^4} + qV$$

$$= \sqrt{(p_x - qA_x)^2c^2 + \left(p_y - qA_y\right)^2c^2 + (p_z - qA_z)^2c^2 + m^2c^4} + qV$$

$$\equiv \sqrt{\left(\vec{p} - q\vec{A}\right)^2c^2 + m^2c^4} + qV$$

$$T = -\frac{mc^2}{\gamma} = -mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

Canonical momentum variable for x axis

$$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = \gamma m\dot{x} + qA_x$$

$$p_y = \frac{\partial\mathcal{L}}{\partial\dot{y}} = \gamma m\dot{y} + qA_y$$

$$p_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = \gamma m\dot{z} + qA_z$$

Equation of motion for charged particle

예제 6-2). 육극자석을 빛의 속도에 가까운 속도로 지나가는 전자의 운동 방정식을 구하시오.

From Wikipedia

$$\vec{A} = \left(0, 0, -\frac{1}{6}\lambda(x^3 - 3xy^2)\right)$$

$$A_z = -\frac{1}{6}\lambda(x^3 - 3xy^2)$$

$$p_x = \gamma m\dot{x}$$

$$p_y = \gamma m\dot{y}$$

$$p_z = \gamma m\dot{z} - qA_z$$

$$\mathcal{H} = \sqrt{\left(\vec{p} - q\vec{A}\right)^2 c^2 + m^2c^4} + qV$$

$$= \sqrt{(p_x)^2c^2 + \left(p_y\right)^2c^2 + (p_z - qA_z)^2c^2 + m^2c^4} \quad \Rightarrow \quad \gamma mc^2$$

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{2}\frac{2p_xc^2}{\gamma mc^2} = \frac{p_x}{\gamma m} \quad \Rightarrow \quad p_x = \gamma m\dot{x}$$

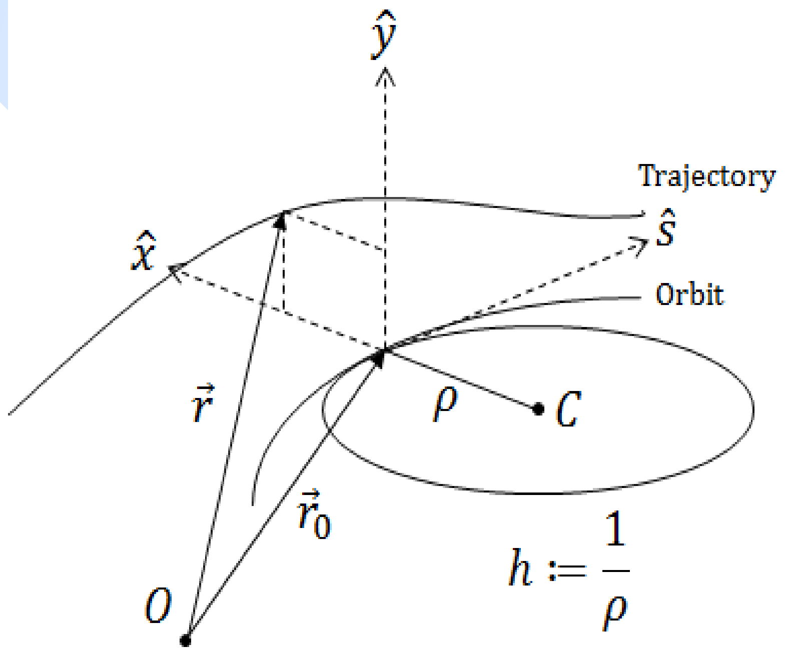
$$\frac{dp_x}{dt} = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{1}{2}\frac{2(p_z - qA_z)c^2}{\gamma mc^2} \times \left(-q\frac{\partial A_z}{\partial x}\right) = \frac{q(p_z - qA_z)}{\gamma m} \times \left(\frac{\partial A_z}{\partial x}\right)$$

$$= \frac{q(\gamma m\dot{z})}{\gamma m} \times \left(-\frac{1}{6}\lambda(3x^2 - 3y^2)\right)$$

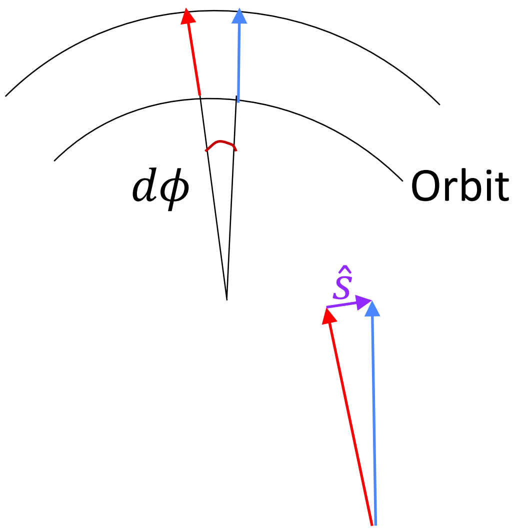
$$= -\frac{q\lambda}{2}(x^2 - y^2)\dot{z} \quad \Rightarrow \quad \dot{\gamma}m\dot{x} + \gamma m\ddot{x} = -\frac{q\lambda}{2}(x^2 - y^2)\dot{z}$$

If $\dot{z} \cong c$, then $\dot{\gamma} \cong 0$ $\Rightarrow$ $\gamma m\ddot{x} = -\frac{q\lambda}{2}(x^2 - y^2)c$

Frenet-Serret coordinate system



$$\mathbf{r} = \mathbf{r}_0 + x\hat{\mathbf{x}} + y\hat{\mathbf{y}}\,.$$

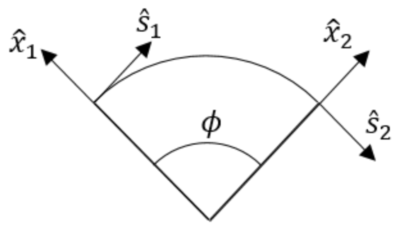


$$\hat{x}_2 = \hat{x}_1 \cos\phi + \hat{s}_1 \sin\phi$$
$$\hat{s}_2 = -\hat{x}_1 \sin\phi + \hat{s}_1 \cos\phi$$

$$\hat{x}_2 = \hat{x}_1 + \hat{s}_1 d\phi$$
$$\hat{s}_2 = -\hat{x}_1 d\phi + \hat{s}_1$$

$$\frac{d\hat{x}}{d\phi} = \hat{s}, \qquad \frac{d\hat{s}}{d\phi} = -\hat{x}$$

$$d\phi = \frac{1}{\rho} ds = hds \rightarrow d\hat{x} = hds \cdot \hat{s}, d\hat{s} = -hds \cdot \hat{x}$$

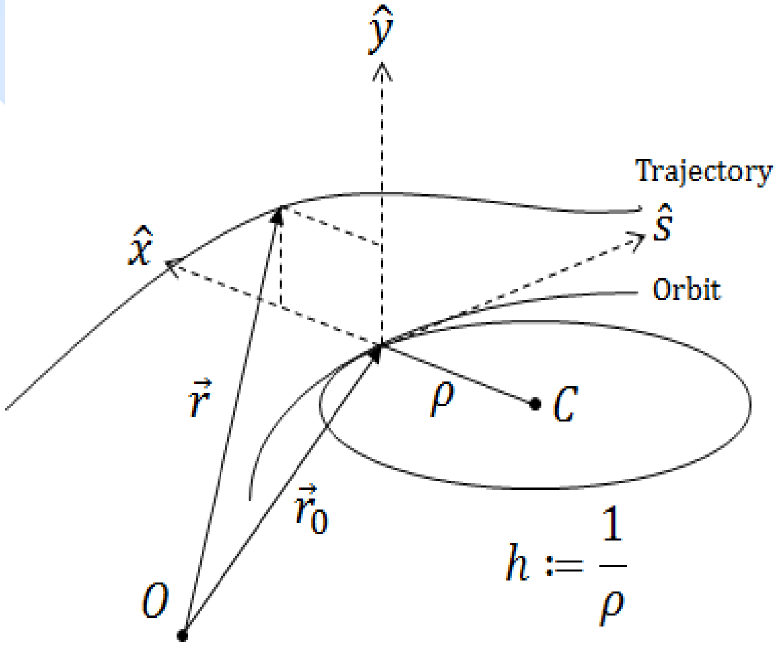


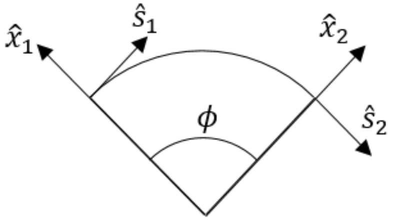
$$d\vec{r} = d\vec{r}_0 + dx\hat{x} + xd\hat{x} + dy\hat{y} + yd\hat{y}$$

$$= ds\hat{s} + dx\hat{x} + hxds\hat{s} + dy\hat{y} \cdots (d\hat{y} = 0)$$

$$= dx\hat{x} + dy\hat{y} + (1 + hx)ds\hat{s}$$

Frenet-Serret coordinate system





$$\mathbf{r} = \mathbf{r}_0 + x\hat{\mathbf{x}} + y\hat{\mathbf{y}}\,.$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + (1 + hx)\dot{s}\hat{s}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + (1 + hx)^2\dot{s}^2$$

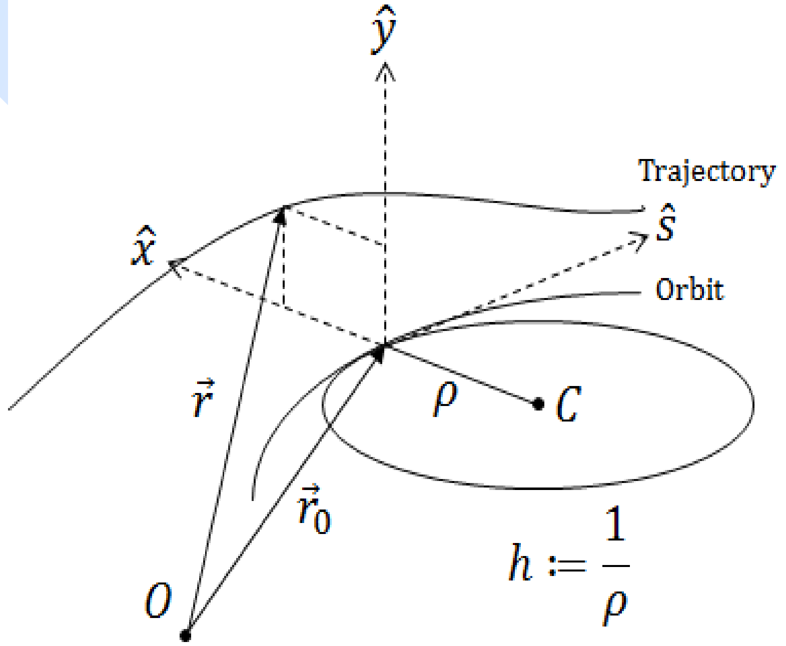
$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + (1 + hx)^2\dot{s}^2)$$

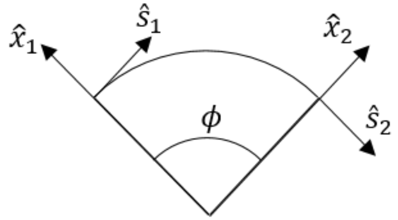
$$0 = \frac{d}{dt}\left(\gamma\frac{\partial}{\partial \dot{q}_j}(T) - \frac{\partial}{\partial \dot{q}_j}(V)\right) - \gamma\frac{\partial}{\partial q_j}(T) + \frac{\partial}{\partial q_j}(V)$$

Parc's equation

$$\simeq \gamma\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}T - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}V - \gamma\frac{\partial}{\partial q_j}T + \frac{\partial}{\partial q_j}V$$

Frenet-Serret coordinate system





$$\gamma\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}T - \frac{\partial}{\partial q_j}T\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}V - \frac{\partial}{\partial q_j}V$$

$$\gamma\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}T - \frac{\partial}{\partial q_j}T\right) = Q_j$$

$$(L.H.S)_x = \gamma\left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}}T - \frac{\partial}{\partial x}T\right)$$

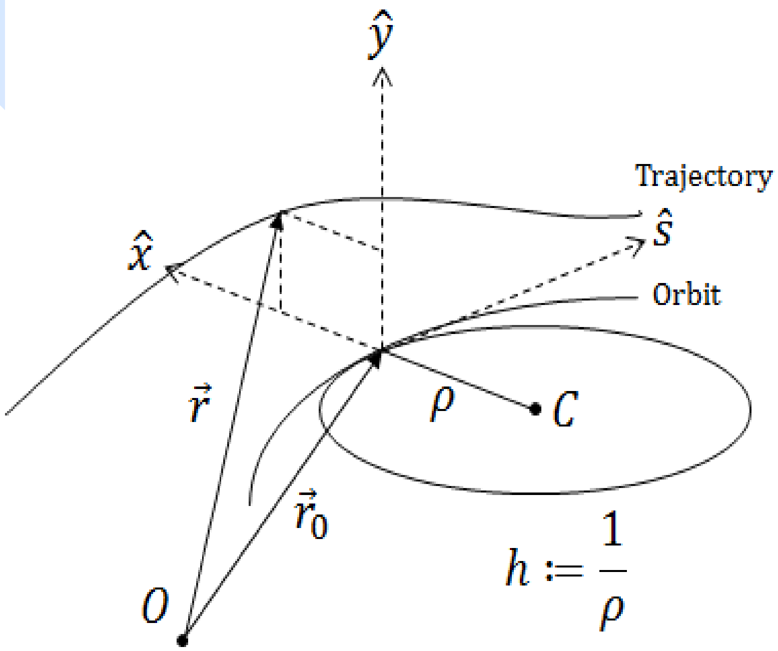
$$= \gamma\left(\frac{d}{dt}(m\dot{x}) - mh(1+hx)\dot{s}^2\right)$$

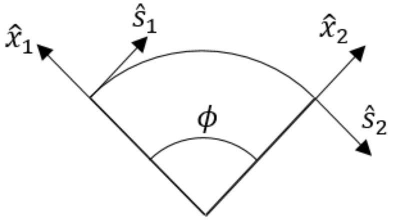
$$= \gamma(m\ddot{x} - mh(1+hx)\dot{s}^2)$$

$$(R.H.S)_x = Q_x = \vec{F}\cdot\frac{\partial \vec{r}}{\partial x} = \left(q\vec{v}\times\vec{B}\right)\cdot\hat{x}$$

$$\gamma(m\ddot{x} - mh(1+hx)\dot{s}^2) = \left(q\vec{v}\times\vec{B}\right)\cdot\hat{x}$$

Frenet-Serret coordinate system





$$(R.H.S)_x$$
$$= \left(q(\dot{x}\hat{x} + \dot{y}\hat{y} + (1 + hx)\dot{s}\hat{s}) \times \left(B_x\hat{x} + B_y\hat{y} + B_s\hat{s}\right)\right) \cdot \hat{x}$$

$$= q\left(\dot{y}B_s - (1 + hx)\dot{s}B_y\right) \qquad B_s = 0$$

$$(R.H.S)_x = -q(1 + hx)\dot{s}B_y$$

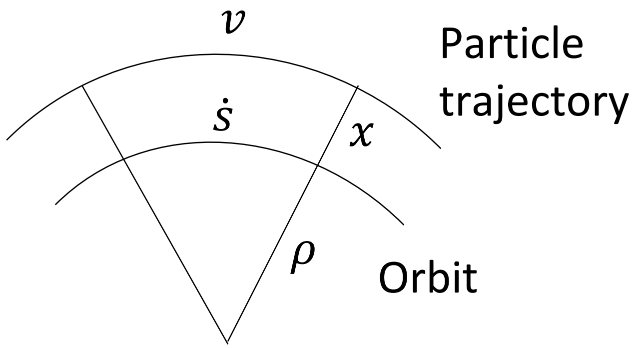
$$\gamma(m\ddot{x} - mh(1 + hx)\dot{s}^2) = -q(1 + hx)\dot{s}B_y$$

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x'\dot{s}$$

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}(x'\dot{s}) = \dot{s}\frac{d}{dt}(x') + x'\frac{d}{dt}(\dot{s}) = \dot{s}^2x'' + x'\ddot{s}$$

$$x''\dot{s}^2 + x'\ddot{s} - h(1 + hx)\dot{s}^2 = -\frac{q}{\gamma m}(1 + hx)\dot{s}B_y$$

Frenet-Serret coordinate system



$$x''\dot{s}^2 + x'\ddot{s} - h(1+hx)\dot{s}^2 = -\frac{q}{\gamma m}(1+hx)\dot{s}B_y$$

$$\ddot{s} \cong 0 \qquad \dot{s} \cong constant$$

$$x'' - h(1+hx) = -\frac{v}{\dot{s}}\frac{q}{\gamma m v}(1+hx)B_y$$

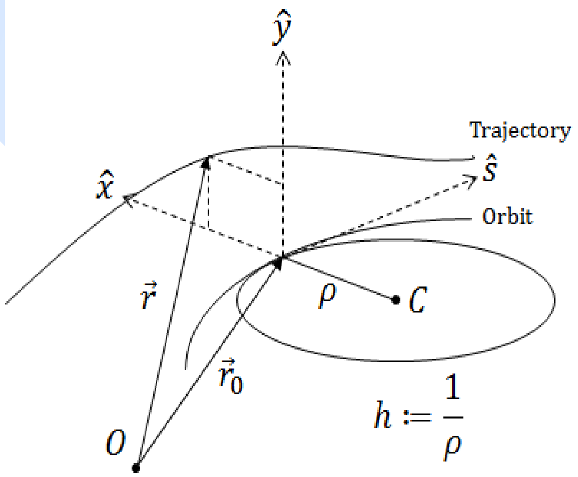
$$x'' - \frac{1}{\rho}\left(1+\frac{x}{\rho}\right) = -\frac{v}{\dot{s}}\frac{q}{p}\left(1+\frac{x}{\rho}\right)B_y$$

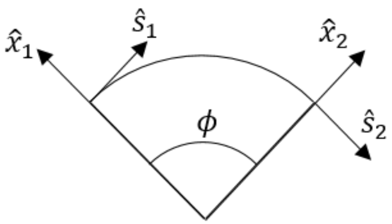
$$x'' - \frac{1}{\rho}\left(1+\frac{x}{\rho}\right) = -\frac{q}{p}\left(1+\frac{x}{\rho}\right)^2 B_y$$

$$v = \dot{s}\frac{\rho + x}{\rho} = \dot{s}\left(1+\frac{x}{\rho}\right)$$

$$\frac{v}{\dot{s}} = 1 + \frac{x}{\rho}$$

Magnetic field





$$\vec{B} = (B_x\,,\quad B_y\,,\quad 0)$$

Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$ $\quad F = qvB_y$

Centrifugal force $F = \dfrac{mv^2}{\rho}$

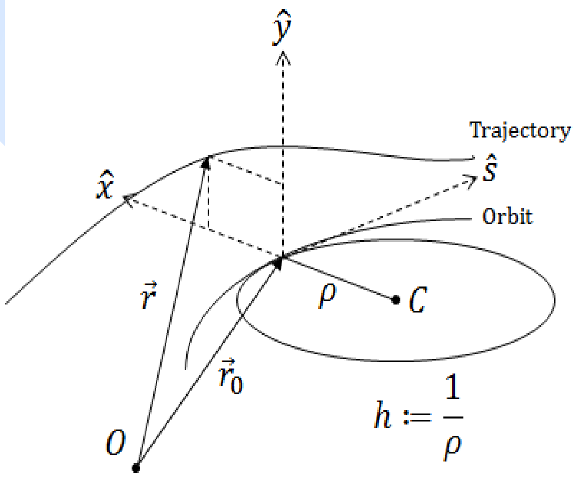
$$\frac{mv^2}{\rho} = qvB_y \;\Rightarrow\; \frac{mv}{\rho} = qB_y \;\Rightarrow\; \frac{p}{\rho} = qB_y \;\Rightarrow\; \frac{1}{\rho} = \frac{q}{p}B_y$$

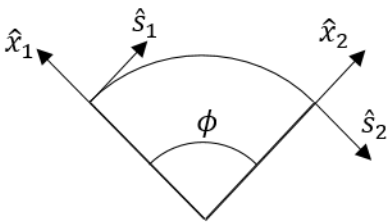
$$B_y(x) = B_y(0) + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_y}{dx^3}x^3 + \cdots$$

$$\frac{q}{p}B_y(x) = \frac{q}{p}B_y(0) + \frac{q}{p}\frac{dB_y}{dx}x + \frac{1}{2!}\frac{q}{p}\frac{d^2B_y}{dx^2}x^2 + \cdots$$

$$= \frac{1}{\rho} + kx + \frac{1}{2!}Sx^2 + \cdots$$

Magnetic field





$$\frac{q}{p}B_y(x) = \frac{q}{p}B_y(0) + \frac{q}{p}\frac{dB_y}{dx}x + \frac{1}{2!}\frac{q}{p}\frac{d^2B_y}{dx^2}x^2 + \cdots$$

$$\frac{q}{p}B_y(x) = \frac{1}{\rho} + kx + \frac{1}{2!}Sx^2 + \cdots$$

Multipole	Definition	effect
dipole	$\frac{1}{\rho} = \frac{q}{p}B_y(0)$	Beam steering
quadrupole	$k = \frac{q}{p}\frac{dB_y}{dx}$	Beam focusing
sextupole	$S = \frac{q}{p}\frac{d^2B_y}{dx^2}$	Chromaticity compensation
etc	…	…

Equation of motion for charged particle in storage ring

$$x'' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) = -\frac{q}{p}\left(1 + \frac{x}{\rho}\right)^2 B_y$$

$$B_y = \frac{1}{\rho} + kx$$

$$x'' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) = -\left(1 + \frac{x}{\rho}\right)^2 \left(\frac{1}{\rho} + kx\right) \cong -\left(1 + \frac{2x}{\rho}\right)\left(\frac{1}{\rho} + kx\right)$$

$$x'' - \frac{1}{\rho} - \frac{x}{\rho^2} \cong -\frac{1}{\rho} - kx - \frac{2x}{\rho^2} - k\frac{2x^2}{\rho}$$

$$x'' - \frac{x}{\rho^2} = -kx - \frac{2x}{\rho^2}$$

$$x'' + \left(k + \frac{1}{\rho^2}\right)x = 0$$

$$x'' + K(s)x = 0$$

Hill's equation of motion

$$K(s) = k(s) + \frac{1}{\rho(s)^2}$$

강의 요약

- Legendre transform

$$g(u, y) = f(x, y) - ux$$

- Energy function

$$h = -\mathcal{L} + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \dot{q}_k$$

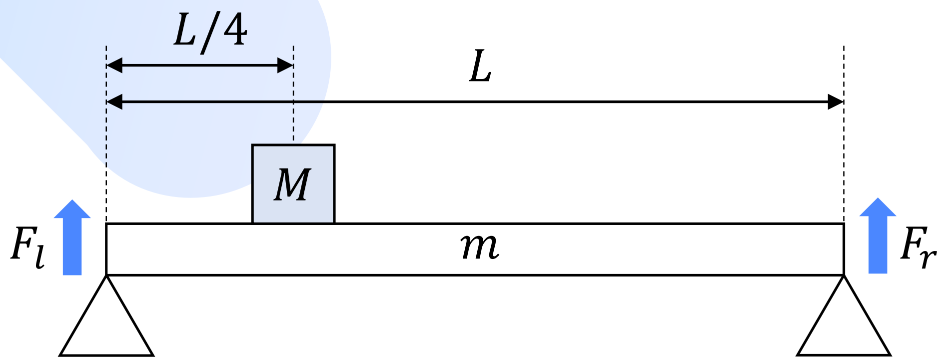
- Hamiltonian

$$\mathcal{H} = \sqrt{\left(\vec{p} - q\vec{A}\right)^2 c^2 + m^2 c^4} + qV$$

- Hill’s equation of motion

$$x'' + K(s)x = 0$$

Quiz) 다음 상황에서 virtual work principle을 적용하여 아래에 주어진 관계식을 유도하시오.



$$\sum F = F_l + F_r - Mg - mg = 0$$

Energy function의 이해

Cartesian 좌표계에서 energy function의 값이 energy와 같다는 것을 상대론이 적용되지 않는
경우와 상대론이 적용되는 경우로 나누어서 보이시오.

$$h = -\mathcal{L} + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q_k}} \dot{q_k} \qquad \mathcal{L}(q, \dot{q}) = T(\dot{q}) - V(q)$$

Relativistic case

$$T = -\frac{mc^2}{\gamma} = -mc^2 \sqrt{1 - \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{c^2}}$$

$$\sum_k \frac{\partial \mathcal{L}}{\partial \dot{q_k}} \dot{q_k} = \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{x} + \frac{\partial \mathcal{L}}{\partial \dot{y}} \dot{y} + \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{z} = \gamma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \gamma m v^2$$

$$h = -\mathcal{L} + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q_k}} \dot{q_k} = -T + V + \gamma m v^2 = -\frac{mc^2}{\gamma} + V + \gamma m v^2 = \gamma m c^2 + V = E$$

숙제
)

$$-\frac{mc^2}{\gamma} + \gamma m v^2 = \gamma m c^2$$