2022 ATE accelerator school

가속기 물리 I

POSTECH

Division of Advanced Nuclear Engineering and Pohang Accelerator Laboratory

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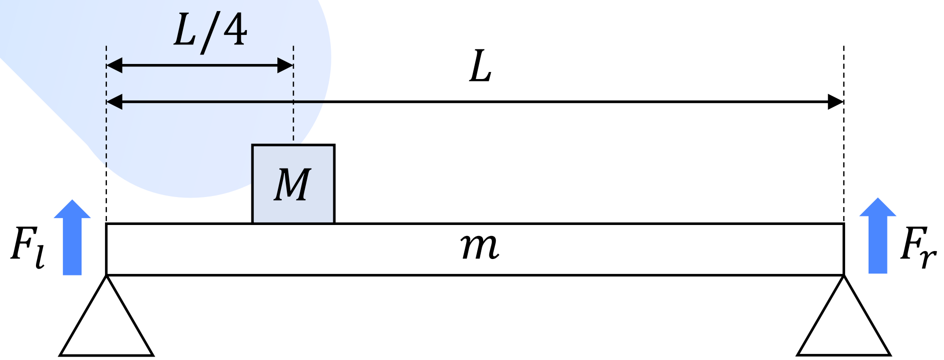
2022.08.08~2022.08.12

Today’s agenda

- 지렛대의 원리(Lever principle)를 증명하고 이해한다.
- 가상의 일의 원리(virtual work principle) 을 이해한다.
- 달랑베르의 원리(D’Alembert principle) 을 이해한다.
- Lagrangian 을 이용하여 운동 방정식을 유도한다.

예제 1: 각 받침대에 가해지는 힘의 크기를 구하시오.

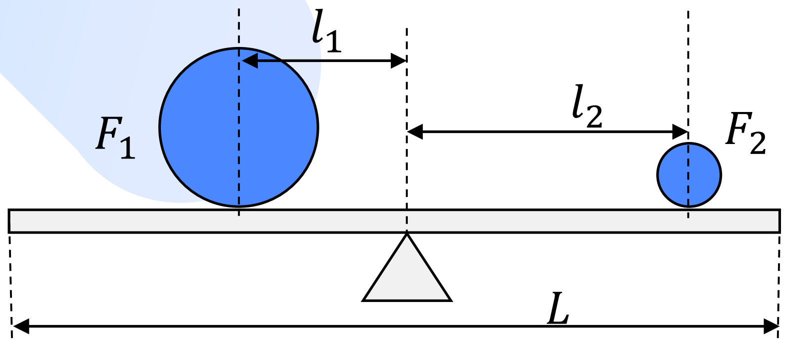


$$\sum F = F_l + F_r - Mg - mg = 0$$

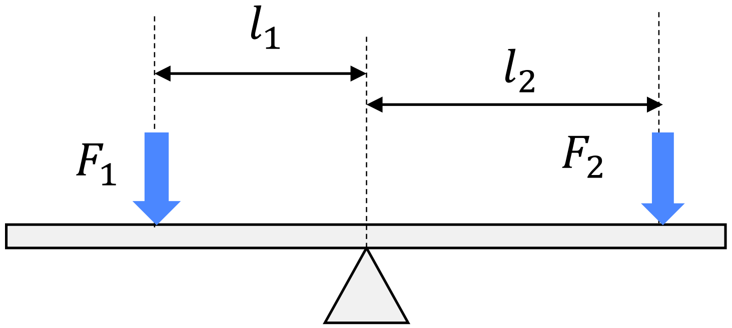
$$\sum \tau = F_l \cdot 0 - Mg \cdot \frac{L}{4} - mg \cdot \frac{L}{2} + F_r \cdot L = 0$$

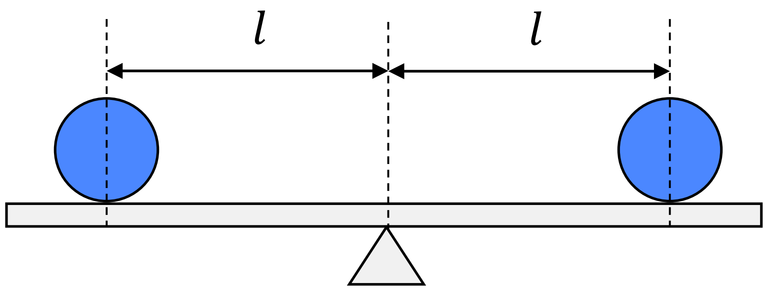
$$F_l = \frac{1}{2}mg + \frac{3}{4}Mg$$
$$F_r = \frac{1}{2}mg + \frac{1}{4}Mg$$

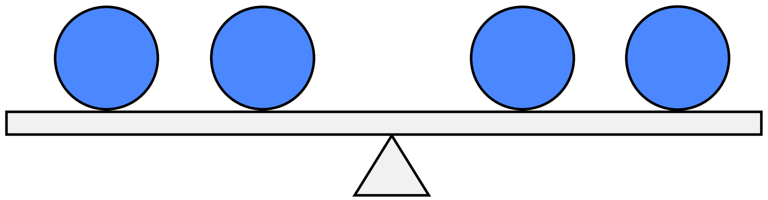
Lever Principle

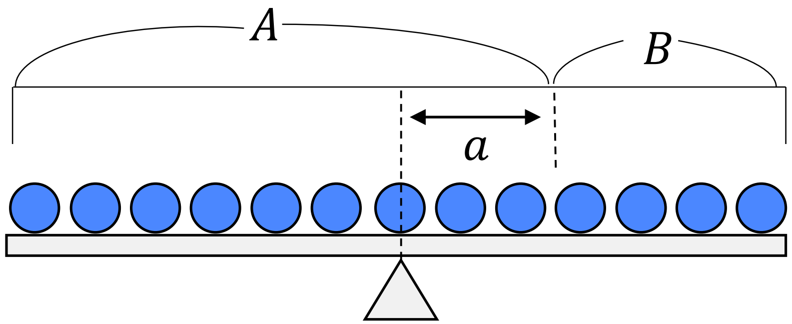


$$F_1 \cdot l_1 = F_2 \cdot l_2$$

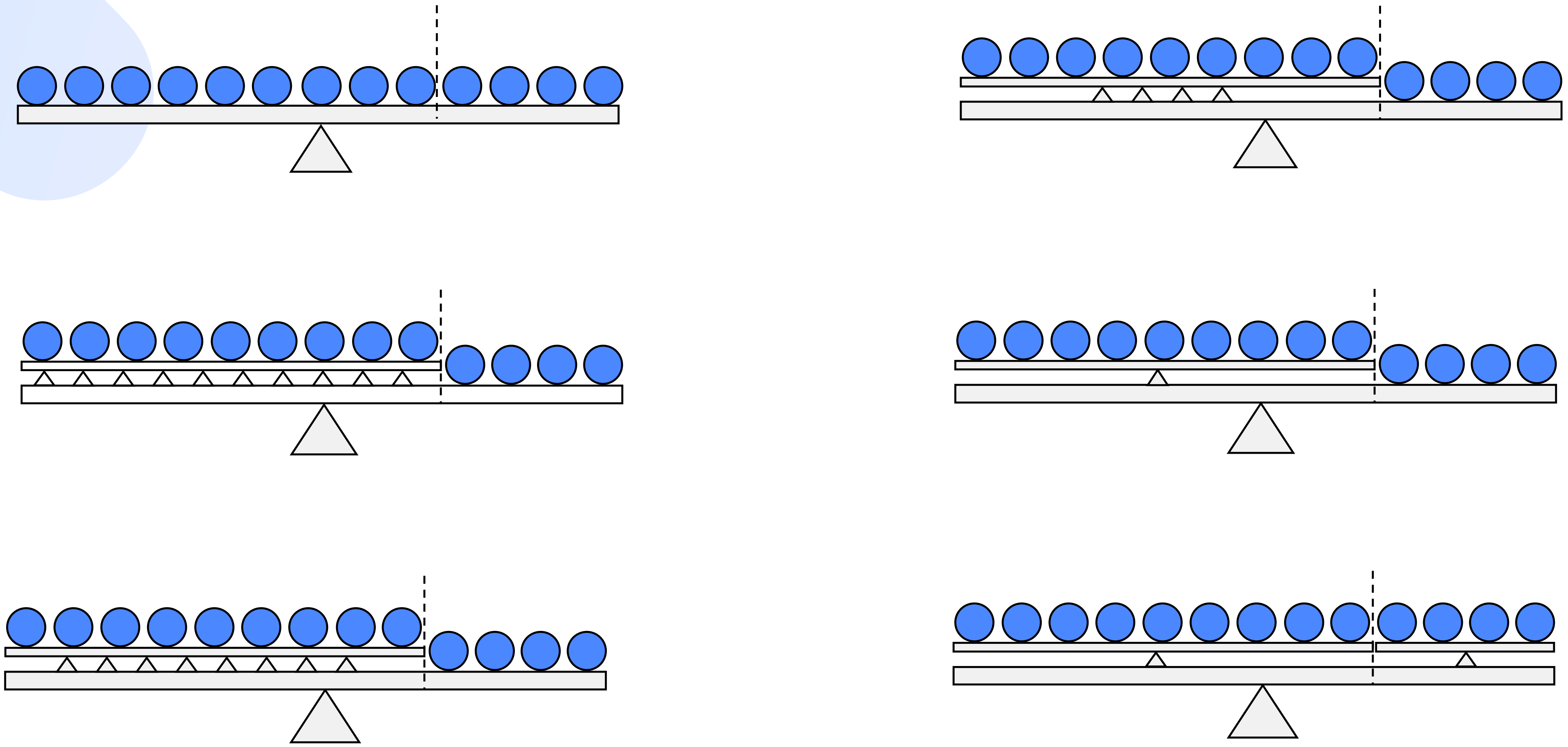




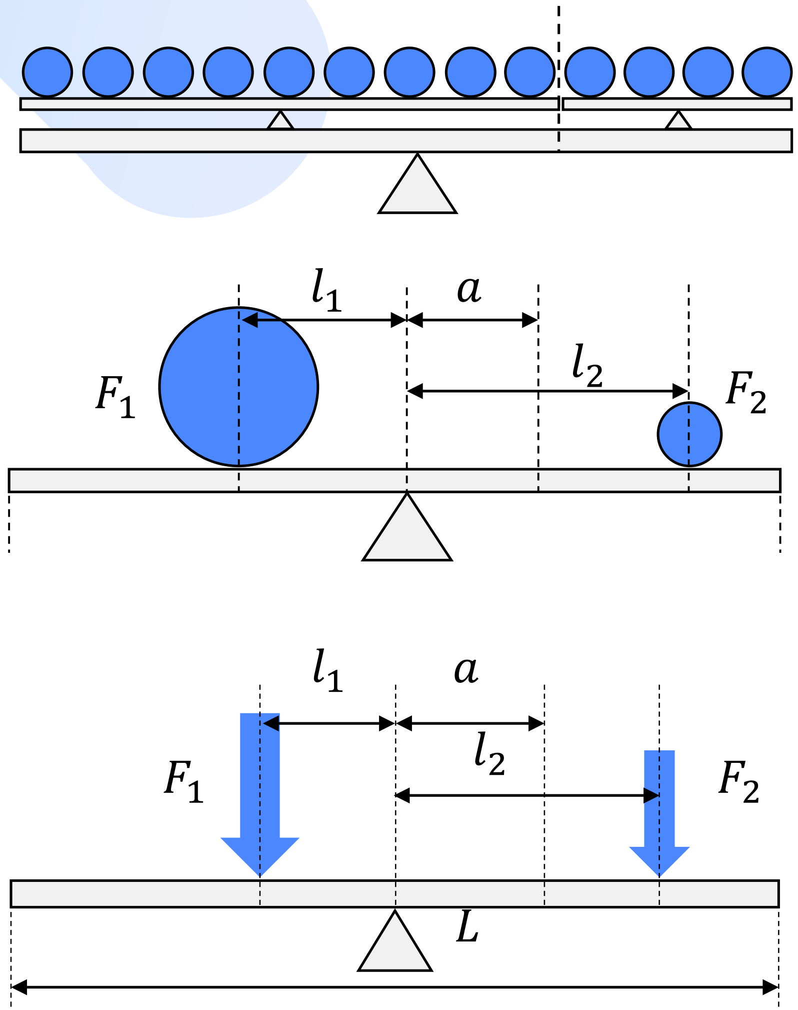




Lever Principle



Lever Principle



$$F_1 = \rho\left(\frac{L}{2}+a\right) \qquad F_2 = \rho\left(\frac{L}{2}-a\right)$$

$$l_1 + a = \frac{1}{2}\left(\frac{L}{2}+a\right) \quad \rightarrow l_1 = \frac{L}{4}-\frac{a}{2}$$

$$l_2 - a = \frac{1}{2}\left(\frac{L}{2}-a\right) \quad \rightarrow l_2 = \frac{L}{4}+\frac{a}{2}$$

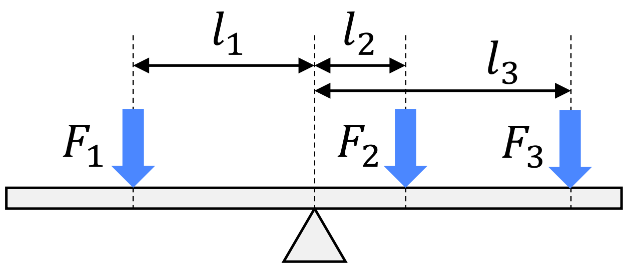
$$\mathrm{F}_1 \times l_1 = F_2 \times l_2 = \frac{\rho}{2}\left(\frac{L}{2}+a\right)\left(\frac{L}{2}-a\right)$$

(힘의 크기) × (영역 중심 위치) = (일정)

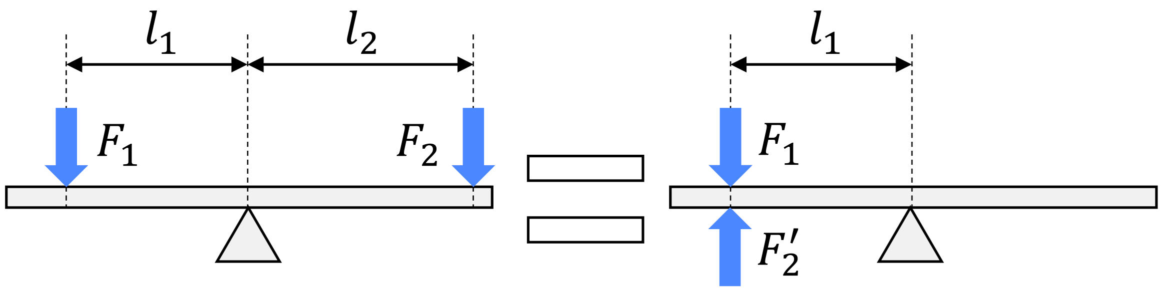
$$F_1 \cdot l_1 = F_2 \cdot l_2$$

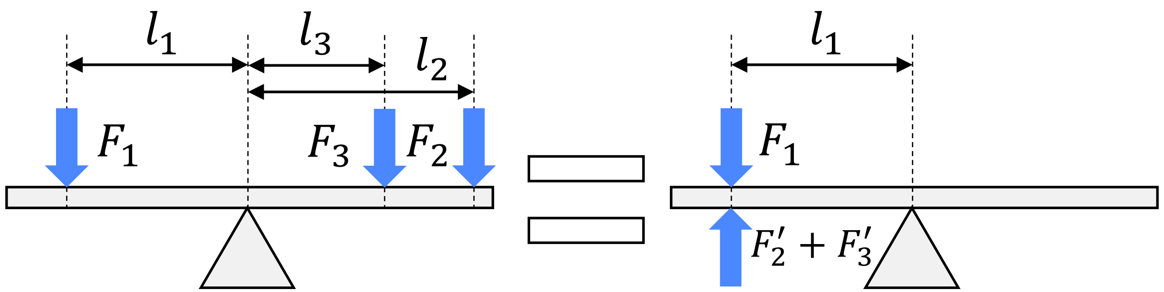
지레대의 원리(Lever Principle)

Lever Principle



$$F_1 l_1 = F_2 l_2 + F_3 l_3$$



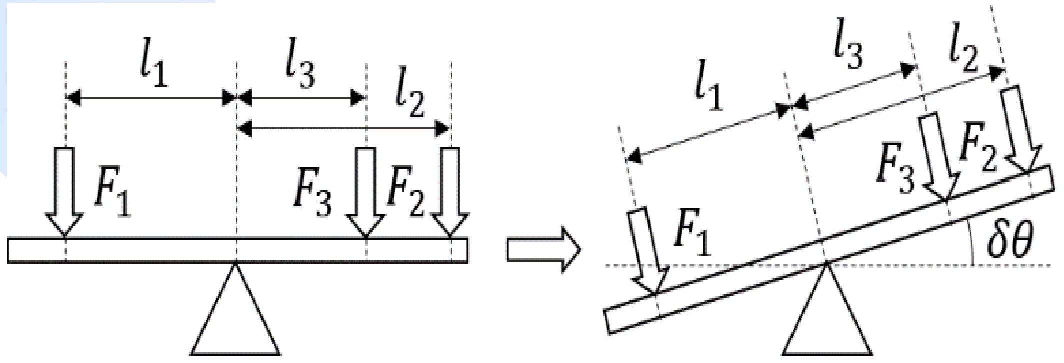


$$F_1 \cdot l_1 = F_2 \cdot l_2$$

$$F_1 = \frac{F_2 l_2}{l_1} \equiv F_2'$$

$$F_1 = F_2' + F_3' = \frac{F_2 l_2}{l_1} + \frac{F_3 l_3}{l_1}$$

Virtual work Principle

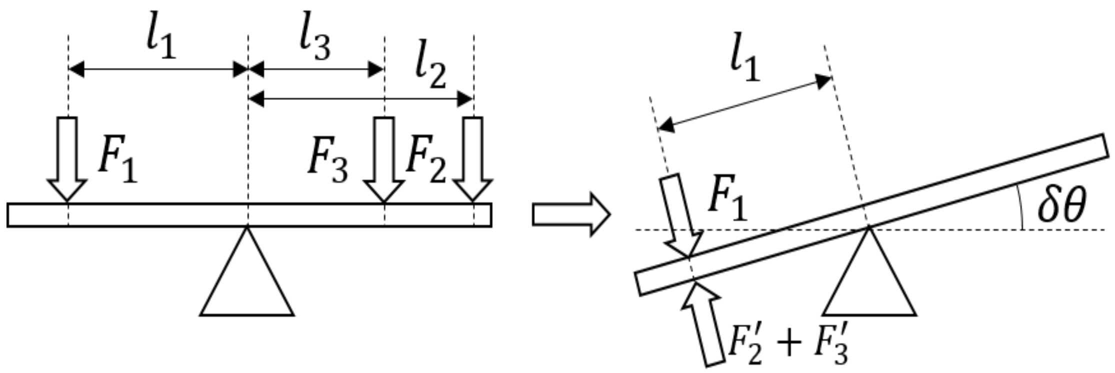


$$W = F \cdot l \qquad \delta W = F \cdot \delta l$$

$$\delta W = F_1 \cdot l_1\delta\theta - F_2 \cdot l_2\delta\theta - F_3 \cdot l_3\delta\theta = 0$$

$$F_1 l_1 = F_2 l_2 + F_3 l_3$$

$$F_1 \cdot \delta\theta - F_2 \cdot \frac{l_2}{l_1}\delta\theta - F_3 \cdot \frac{l_3}{l_1}\delta\theta = 0$$



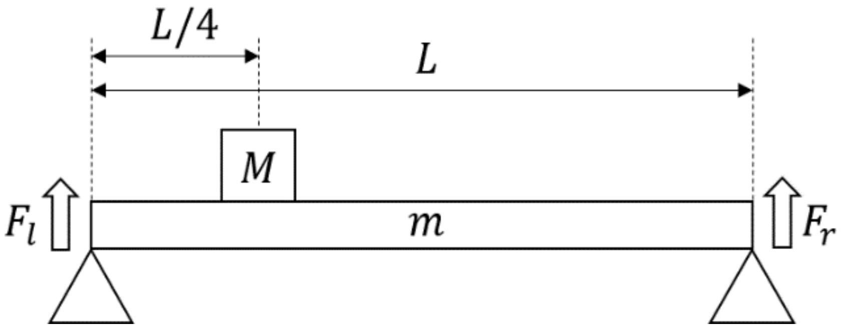
$$F_1 \cdot \delta\theta - F_2{}' \cdot \delta\theta - F_3{}' \cdot \delta\theta = 0$$

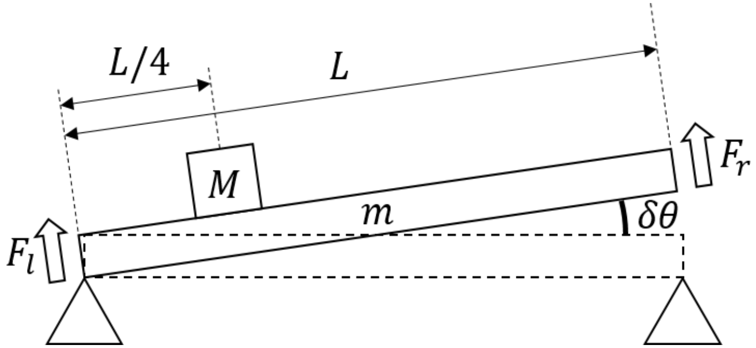
$$(F_1 - F_2{}' - F_3{}') \cdot \delta\theta = 0$$

$$F_1 = F_2' + F_3' \quad \Rightarrow \quad F_1 l_1 = F_2 l_2 + F_3 l_3$$

예제1

Virtual work Principle





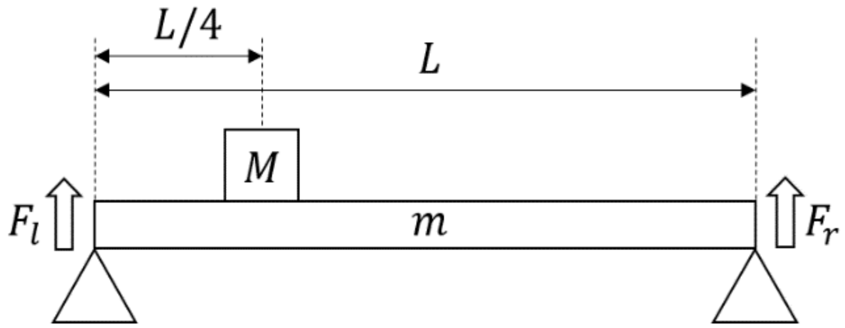
$$F_l \cdot 0 \ \cdot \delta\theta + F_r \cdot L \cdot \delta\theta - Mg \cdot \frac{L}{4} \cdot \delta\theta - mg \cdot \frac{L}{2} \cdot \delta\theta = 0$$

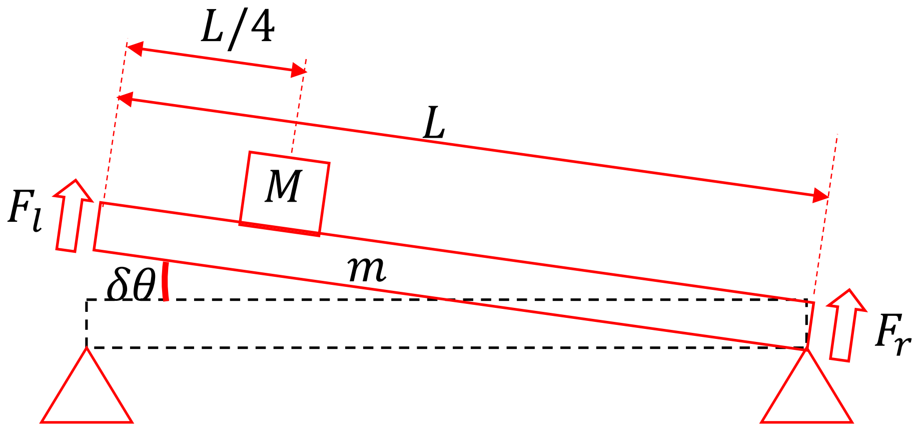
$$F_r = \frac{1}{4}Mg + \frac{1}{2}mg$$

예제1) 위 구조물에서 오른쪽 받침대와 왼쪽 받침대에 가해지는 힘을 각각 구하시오.

예제1

Virtual work Principle





예제1) 위 구조물에서 오른쪽 받침대와 왼쪽 받침대에 가해지는 힘을 각각 구하시오.

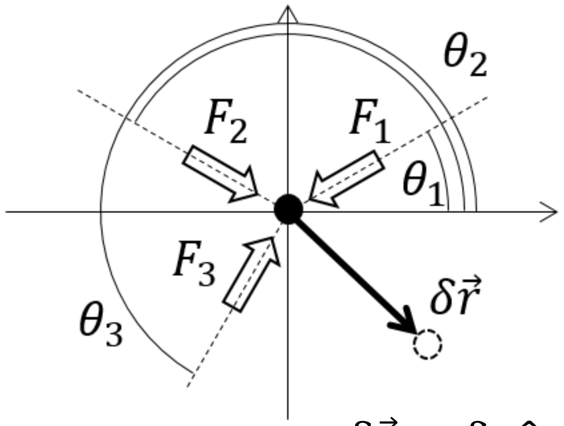
$$F_l \cdot L \cdot \delta\theta + F_r \cdot 0 \cdot \delta\theta - Mg \cdot \frac{3L}{4} \cdot \delta\theta - mg \cdot \frac{L}{2} \cdot \delta\theta = 0$$

$$F_l = \frac{3}{4}Mg + \frac{1}{2}mg$$

Virtual work principle in Mechanics

Virtual work principle을 운동하는 입자 한 개에 적용하기 위한 생각의 전환

<u>첫 번째 생각의 전환</u>:



$$\delta\vec{r} = \delta x\hat{x} + \delta y\hat{y}$$

$$(F_1\cos\theta_1 + F_2\cos\theta_2 + F_3\cos\theta_3)\delta x = 0$$

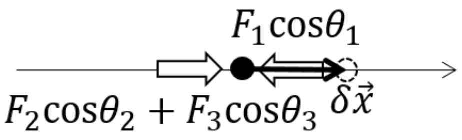
$$(F_1\sin\theta_1 + F_2\sin\theta_2 + F_3\sin\theta_3)\delta y = 0$$

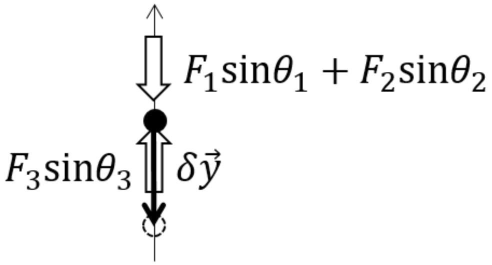
$$\left(\vec{F_1}\cdot\hat{x} + \vec{F_2}\cdot\hat{x} + \vec{F_3}\cdot\hat{x}\right)\delta x + \left(\vec{F_1}\cdot\hat{y} + \vec{F_2}\cdot\hat{y} + \vec{F_3}\cdot\hat{y}\right)\delta y = 0$$

$$\left(\vec{F_1} + \vec{F_2} + \vec{F_3}\right)\cdot\hat{x}\delta x + \left(\vec{F_1} + \vec{F_2} + \vec{F_3}\right)\cdot\hat{y}\delta y = 0$$

$$\left(\vec{F_1} + \vec{F_2} + \vec{F_3}\right)\cdot\delta\vec{r} = 0 \quad \Rightarrow \quad \vec{\boldsymbol{F}}\cdot\delta\vec{r} = 0$$

$$\vec{\boldsymbol{F}} = \vec{F_1} + \vec{F_2} + \vec{F_3} = 0$$





Virtual work principle in Mechanics

동역학에서는 힘이 균형을 이루지 않기 때문에 입자가 가속을 한다.

Virtual work principle on a particle

$$\vec{\boldsymbol{F}} \cdot \delta\vec{r} = 0$$

$$\vec{\boldsymbol{F}} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = 0$$

두 번째 생각의 전환:

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \vec{F} \neq 0$$

$$\vec{F} - \frac{d\vec{p}}{dt} = 0 \qquad \frac{d\vec{p}}{dt} \equiv \vec{F}_I$$

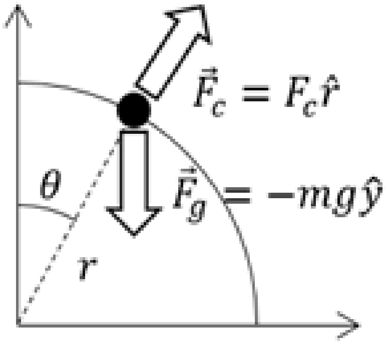
$$\vec{F} - \vec{F}_I = 0$$

$$\left(\vec{F} - \frac{d\vec{p}}{dt}\right) \cdot \delta\vec{r} = 0$$

달랑베르의 원리(D'Alembert's principle)

예제2

예제2) 물체의 위치에 따른 구속력을 구하시오.



$$\vec{r} = x\hat{x} + y\hat{y}$$
$$x = r\sin\theta$$
$$y = r\cos\theta$$

$$\hat{r} = \sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$
$$\hat{\theta} = \cos\theta\,\hat{x} - \sin\theta\,\hat{y}$$
$$\hat{x} = \cos\theta\,\hat{\theta} + \sin\theta\,\hat{r}$$
$$\hat{y} = -\sin\theta\,\hat{\theta} + \cos\theta\,\hat{r}$$
$$\vec{r} = x\hat{x} + y\hat{y} = r\hat{r}$$

$$\left(\vec{F} - \frac{d\vec{p}}{dt}\right)\cdot\delta\vec{r} = 0$$

$$\vec{F}\cdot\delta\vec{r} - \frac{d\vec{p}}{dt}\cdot\delta\vec{r} = 0$$

$$\delta\vec{r} = \delta x\hat{x} + \delta y\hat{y}$$

$$\delta x = \delta r\sin\theta + r\delta\theta\cos\theta$$
$$\delta y = \delta r\cos\theta - r\delta\theta\sin\theta$$
$$\delta\vec{r} = \delta r\hat{r} + r\delta\theta\hat{\theta}$$

$$\vec{F}\cdot\delta\vec{r} = (\vec{F}\cdot\hat{r})\delta r + (r\vec{F}\cdot\hat{\theta})\delta\theta$$

$$\vec{F}\cdot\hat{r} = -mg\cos\theta + F_c$$

$$r\vec{F}\cdot\hat{\theta} = mgr\sin\theta$$

$$\vec{F}\cdot\delta\vec{r} = (-mg\cos\theta + F_c)\delta r + (mgr\sin\theta)\delta\theta$$

예제2

$$\frac{d\vec{p}}{dt}\cdot\delta\vec{r} = m\ddot{\vec{r}}\cdot\left(\delta r\hat{r} + r\delta\theta\hat{\theta}\right)$$
$$= \left(m\ddot{\vec{r}}\cdot\hat{r}\right)\delta r + r\left(m\ddot{\vec{r}}\cdot\hat{\theta}\right)\delta\theta$$

$$m\ddot{\vec{r}}\cdot\hat{r} = \frac{d}{dt}\left(m\dot{\vec{r}}\cdot\hat{r}\right) - m\dot{\vec{r}}\frac{d}{dt}(\hat{r})$$

$$= \frac{d}{dt}(m\dot{r}) - m\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)\cdot\left(\dot{\theta}\hat{\theta}\right)$$

$$= m\ddot{r} - mr\dot{\theta}^2$$

$$m\ddot{\vec{r}}\cdot\hat{\theta} = \frac{d}{dt}\left(m\dot{\vec{r}}\cdot\hat{\theta}\right) - m\dot{\vec{r}}\frac{d\hat{\theta}}{dt}$$

$$= \frac{d}{dt}\left(m\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)\cdot\hat{\theta}\right) - m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})\cdot\left(-\dot{\theta}\hat{r}\right)$$

$$= m\dot{r}\dot{\theta} + mr\ddot{\theta} + m\dot{r}\dot{\theta} = mr\ddot{\theta} + 2m\dot{r}\dot{\theta}$$

$$\frac{d\vec{p}}{dt}\cdot\delta\vec{r} = \left(m\ddot{r} - mr\dot{\theta}^2\right)\delta r + (mr\ddot{\theta} + 2m\dot{r}\dot{\theta})\delta\theta$$

$$\vec{F}\cdot\delta\vec{r} = (-mg\cos\theta + F_c)\delta r + (mgr\sin\theta)\delta\theta$$

$$\hat{\theta} : 0 = mr\left[g\sin\theta + r\ddot{\theta} + 2\dot{r}\dot{\theta}\right]$$

$$\hat{r} : F_c = m\left[\ddot{r} + g\cos\theta - r\dot{\theta}^2\right]$$

달랑베르 원리 적용의 일반화

$$\vec{r} = \vec{r}(q_1, q_2, \cdots, q_n, t)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \sum_k \frac{\partial \vec{r}}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}}{\partial t}$$

$$\delta \vec{r} = \sum_j \frac{\partial \vec{r}}{\partial q_j} \delta q_j$$

$$\frac{d\vec{p}}{dt} \cdot \delta \vec{r} = \sum_j m\ddot{\vec{r}} \frac{\partial \vec{r}}{\partial q_j} \delta q_j$$

$$m\ddot{\vec{r}} \frac{\partial \vec{r}}{\partial q_j} \delta q_j = \frac{d}{dt}\left(m\dot{\vec{r}} \frac{\partial \vec{r}}{\partial q_j} \right) \delta q_j - m\dot{\vec{r}} \frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_j} \right) \delta q_j$$

$$= \frac{d}{dt}\left(m\vec{v} \frac{\partial \vec{r}}{\partial q_j} \right) \delta q_j - m\vec{v} \frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_j} \right) \delta q_j$$

달랑베르 원리 적용의 일반화

$$\frac{d}{dt}\frac{\partial \vec{r}}{\partial q_j} = \frac{\partial}{\partial q_j}\frac{d\vec{r}}{dt} = \frac{\partial \vec{v}}{\partial q_j}$$

$$\frac{\partial \vec{v}}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\frac{d\vec{r}}{dt} = \frac{\partial}{\partial \dot{q}_j}\left(\sum_k \frac{\partial \vec{r}}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}}{\partial t}\right) = \frac{\partial \vec{r}}{\partial q_j}$$

$$\frac{d\vec{p}}{dt}\delta\vec{r} = \sum_j \left[\frac{d}{dt}\left(m\vec{v}\frac{\partial \vec{r}}{\partial q_j}\right) - m\vec{v}\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_j}\right)\right]\cdot \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(m\vec{v}\frac{\partial \vec{v}}{\partial \dot{q}_j}\right) - m\vec{v}\frac{\partial \vec{v}}{\partial q_j}\right]\cdot \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m\vec{v}^2\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m\vec{v}^2\right)\right]\cdot \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T)\right) - \frac{\partial}{\partial q_j}(T)\right]\cdot \delta q_j$$

달랑베르 원리 적용의 일반화

$$\vec{F}\cdot\delta\vec{r} = \sum_j \vec{F}\cdot\frac{\partial\vec{r}}{\partial q_j}\delta q_j$$

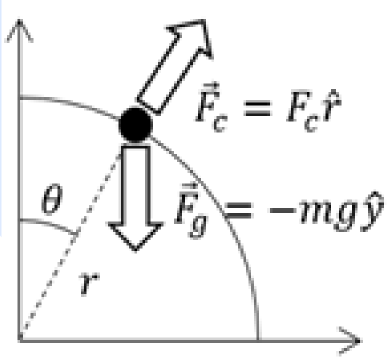
$$= \sum_{\mathrm{j}} Q_j \delta q_j$$

$$\left(\vec{F} - \frac{d\vec{p}}{dt}\right)\cdot\delta\vec{r} = \sum_j \left[Q_j - \frac{d}{dt}\frac{\partial}{\partial\dot{q}_j}(T) + \frac{\partial}{\partial q_j}(T)\right]\delta q_j = 0$$

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial\dot{q}_j}(T) - \frac{\partial}{\partial q_j}(T)$$

: Lagrange 방정식

예제 2의 다른 풀이



$$\vec{r} = x\hat{x} + y\hat{y}$$

$$x = r\sin\theta$$

$$y = r\cos\theta$$

$$\hat{r} = \sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\hat{\theta} = \cos\theta\,\hat{x} - \sin\theta\,\hat{y}$$

$$\hat{x} = \cos\theta\,\hat{\theta} + \sin\theta\,\hat{r}$$

$$\hat{y} = -\sin\theta\,\hat{\theta} + \cos\theta\,\hat{r}$$

$$\vec{r} = x\hat{x} + y\hat{y} = r\hat{r} \quad \Rightarrow \quad \frac{\partial\vec{r}}{\partial r} = \hat{r}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$\vec{F} = F_c\hat{r} - mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}$$

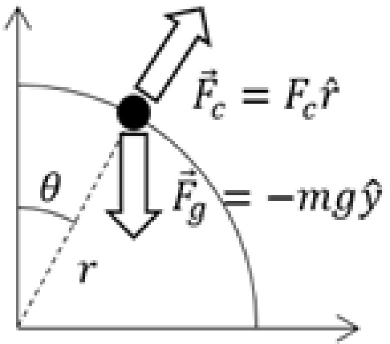
$$Q_r = \vec{F}\frac{\partial\vec{r}}{\partial r} = \vec{F}\cdot\hat{r} = F_c - mg\cos\theta$$

$$Q_r = \frac{d}{dt}\frac{\partial}{\partial\dot{r}}(T) - \frac{\partial}{\partial r}(T)$$

$$F_c - mg\cos\theta = m\ddot{r} - mr\dot{\theta}^2$$

$$\therefore F_c = m\left(\ddot{r} + g\cos\theta - r\dot{\theta}^2\right)$$

숙제



$$Q_\theta = \frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}(T) - \frac{\partial}{\partial\theta}(T)$$

$$Q_\theta = \vec{F}\frac{\partial\vec{r}}{\partial\theta} = \vec{F}\frac{\partial(r\sin\theta\,\hat{x} + r\cos\theta\,\hat{y})}{\partial\theta} = r\vec{F}\cdot\hat{\theta} = -mgr\sin\theta$$

$$-mgr\sin\theta = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\therefore\ 0 = mr\left(g\sin\theta + r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

상대성 이론이 적용된 라그랑지 방정식

D'Alembert's principle

$$\left(\vec{F} - \frac{d\vec{p}}{dt}\right) \cdot \delta\vec{r} = 0$$

$$\vec{F} \cdot \delta\vec{r} = \sum_j \vec{F} \cdot \frac{\partial\vec{r}}{\partial q_j}\delta q_j$$

$$= \sum_{\mathrm{j}} Q_j \delta q_j$$

$$Q_j = \vec{F} \cdot \frac{\partial\vec{r}}{\partial q_j} = -(\nabla V) \cdot \frac{\partial\vec{r}}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\vec{F} \cdot \delta\vec{r} = -\sum_{\mathrm{j}} \frac{\partial V}{\partial q_j}\delta q_j$$

$$\vec{p} = \gamma m\vec{v} = \frac{1}{\sqrt{1-\beta^2}} m\vec{v} \ldots \left(\beta = \frac{|\vec{v}|}{c}\right)$$

$$\delta\vec{r} = \sum_j \frac{\partial\vec{r}}{\partial q_j}\delta q_j$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \sum_k \frac{\partial\vec{r}}{\partial q_k}\dot{q}_k + \frac{\partial\vec{r}}{\partial t}$$

$$\frac{\partial\vec{v}}{\partial\dot{q}_j} = \frac{\partial\vec{r}}{\partial q_j}$$

$$\frac{d\vec{p}}{dt} \cdot \delta\vec{r} = \frac{d(\gamma m\vec{v})}{dt} \cdot \delta\vec{r}$$

$$= \sum_j \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\beta^2}}\right) \cdot \left(\frac{\partial\vec{r}}{\partial q_j}\right)\delta q_j$$

상대성 이론이 적용된 라그랑지 방정식

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\beta^2}}\right)\cdot\left(\frac{\partial\vec{r}}{\partial q_j}\right)$$

$$=\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\beta^2}}\cdot\frac{\partial\vec{r}}{\partial q_j}\right)-\frac{m\vec{v}}{\sqrt{1-\beta^2}}\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}}{\partial q_j}\right)$$

$$=\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\beta^2}}\cdot\frac{\partial\vec{v}}{\partial \dot{q}_j}\right)-\frac{m\vec{v}}{\sqrt{1-\beta^2}}\cdot\frac{\partial\vec{v}}{\partial q_j}$$

$$=mc^2\left[\frac{d}{dt}\left(\frac{\vec{\beta}}{\sqrt{1-\beta^2}}\cdot\frac{\partial\vec{\beta}}{\partial \dot{q}_j}\right)-\frac{\vec{\beta}}{\sqrt{1-\beta^2}}\cdot\frac{\partial\vec{\beta}}{\partial q_j}\right]$$

$$=mc^2\left[-\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sqrt{1-\beta^2}\right)+\frac{\partial}{\partial q_j}\left(\sqrt{1-\beta^2}\right)\right]$$

$$=\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(-\frac{mc^2}{\gamma}\right)-\frac{\partial}{\partial q_j}\left(-\frac{mc^2}{\gamma}\right)\quad=\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}T-\frac{\partial}{\partial q_j}T\qquad T\equiv-\frac{mc^2}{\gamma}$$

상대성 이론이 적용된 라그랑지 방정식

$$\vec{F}\cdot\delta\vec{r} = -\sum_{\mathrm{j}} \frac{\partial V}{\partial q_j}\delta q_j$$

$$\frac{d\vec{p}}{dt}\cdot\delta\vec{r} = \sum_j \left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}T - \frac{\partial}{\partial q_j}T\right)\delta q_j$$

$$\vec{F}\cdot\delta\vec{r} - \frac{d\vec{p}}{dt}\cdot\delta\vec{r} = 0$$

$$-\frac{\partial V}{\partial q_j} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}T + \frac{\partial}{\partial q_j}T = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}T - \frac{\partial}{\partial q_j}(T-V) = 0$$

$$0 = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-V) - \frac{\partial}{\partial q_j}(T-V)\ \ldots\left(T \equiv -\frac{mc^2}{\gamma}\right)$$

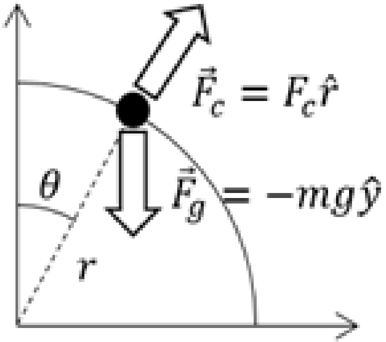
$$= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(\mathcal{L}) - \frac{\partial}{\partial q_j}(\mathcal{L})\ \ldots(\mathcal{L} \equiv T - V)$$

$$K = \gamma mc^2 - mc^2 = mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

$$\neq T = -\frac{mc^2}{\gamma} = -mc^2\sqrt{1-\frac{v^2}{c^2}}$$

상대론 적용된 경우의 라그랑지안은 운동에너지와 위치에너지의 차이가 아니다.

예제 3



$$\mathcal{L} \equiv T - V = -mc^2\sqrt{1 - \frac{a^2\dot{\theta}^2}{c^2}} - mga\cos(\theta) \qquad T = -\frac{mc^2}{\gamma} = -mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

$$\frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}(\mathcal{L}) - \frac{\partial}{\partial\theta}(\mathcal{L}) = 0$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$r = a$$

$$v^2 = a^2\dot{\theta}^2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = -mc^2 \cdot \frac{1}{2} \cdot \gamma \cdot \left(-2\frac{a^2}{c^2}\dot{\theta}\right) = a^2 m\gamma\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}(\mathcal{L}) = a^2 m\dot{\gamma}\dot{\theta} + a^2 m\gamma\ddot{\theta}$$

$$\frac{\partial}{\partial\theta}(\mathcal{L}) = mga\sin(\theta)$$

$$\frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}(\mathcal{L}) - \frac{\partial}{\partial\theta}(\mathcal{L}) = a^2 m\dot{\gamma}\dot{\theta} + a^2 m\gamma\ddot{\theta} - mga\sin(\theta) = 0$$

Another form of the equation of motion

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\beta^2}}\right)\cdot\left(\frac{\partial\vec{r}}{\partial q_j}\right)$$

$$=\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\beta^2}}\cdot\frac{\partial\vec{v}}{\partial\dot{q}_j}\right)-\frac{m\vec{v}}{\sqrt{1-\beta^2}}\cdot\frac{\partial\vec{v}}{\partial q_j}$$

$$=\frac{d}{dt}\left(\gamma m\vec{v}\cdot\frac{\partial\vec{v}}{\partial\dot{q}_j}\right)-\gamma m\vec{v}\cdot\frac{\partial\vec{v}}{\partial q_j}$$

$$=\frac{d}{dt}\left(\gamma\frac{\partial}{\partial\dot{q}_j}\left(\frac{1}{2}mv^2\right)\right)-\gamma\frac{\partial}{\partial q_j}\left(\frac{1}{2}mv^2\right)$$

$$=\frac{d}{dt}\left(\gamma\frac{\partial T}{\partial\dot{q}_j}\right)-\gamma\frac{\partial T}{\partial q_j}$$

$$\vec{F}\cdot\delta\vec{r}=-\sum_{\mathrm{j}}\frac{\partial V}{\partial q_j}\delta q_j=\sum_{\mathrm{j}}Q_j\delta q_j$$

$$\frac{d\vec{p}}{dt}\cdot\delta\vec{r}=\sum_j\left(\frac{d}{dt}\left(\gamma\frac{\partial T}{\partial\dot{q}_j}\right)-\gamma\frac{\partial T}{\partial q_j}\right)\delta q_j$$

$$\vec{F}\cdot\delta\vec{r}-\frac{d\vec{p}}{dt}\cdot\delta\vec{r}=0$$

$$-\frac{\partial V}{\partial q_j}-\frac{d}{dt}\left(\gamma\frac{\partial T}{\partial\dot{q}_j}\right)+\gamma\frac{\partial T}{\partial q_j}=0$$

$$\frac{d}{dt}\left(\gamma\frac{\partial T}{\partial\dot{q}_j}\right)+\gamma\frac{\partial T}{\partial q_j}-\frac{\partial V}{\partial q_j}=0$$

$$\frac{d}{dt}\left(\gamma\frac{\partial T}{\partial\dot{q}_j}-\frac{\partial V}{\partial\dot{q}_j}\right)+\gamma\frac{\partial T}{\partial q_j}-\frac{\partial V}{\partial q_j}=0$$ Parc's equation

하전 입자의 운동 방정식

Newton equation

$$\frac{d\vec{p}}{dt} = \vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k$$

$$Q_k = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_k} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \frac{\partial \vec{r}}{\partial q_k}$$

$$Q_k = q\left(-\nabla V - \frac{\partial}{\partial t}\vec{A} + \vec{v} \times \nabla \times \vec{A}\right) \cdot \frac{\partial \vec{r}}{\partial q_k}$$

Maxwell equations

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho$$

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \nabla \times \left(\vec{E} + \frac{\partial}{\partial t}\vec{A}\right) = 0 \quad \Rightarrow \quad \vec{E} = -\nabla V - \frac{\partial}{\partial t}\vec{A}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

하전 입자의 운동 방정식

Cartesian coordinate

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = Q_x$$

$$Q_x = \vec{F} \cdot \frac{\partial \vec{r}}{\partial x} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \frac{\partial \vec{r}}{\partial x}$$

$$Q_x = q\left(-\nabla V - \frac{\partial}{\partial t}\vec{A} + \vec{v} \times \nabla \times \vec{A}\right) \cdot \frac{\partial \vec{r}}{\partial x}$$

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{s} \equiv v_x\hat{x} + v_y\hat{y} + v_s\hat{s}$$

$$\vec{A} = \left(\vec{A} \cdot \hat{x}\right)\hat{x} + \left(\vec{A} \cdot \hat{y}\right)\hat{y} + \left(\vec{A} \cdot \hat{z}\right)\hat{z} \equiv A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\frac{\partial \vec{r}}{\partial x} = \frac{\partial(x\hat{x} + y\hat{y} + z\hat{z})}{\partial x} = \hat{x}$$

$$Q_x = q\left(-\nabla V - \frac{\partial}{\partial t}\vec{A} + \vec{v} \times \nabla \times \vec{A}\right) \cdot \hat{x}$$

$$= q\left(-\nabla V \cdot \hat{x} - \frac{\partial}{\partial t}\vec{A} \cdot \hat{x} + \left(\vec{v} \times \nabla \times \vec{A}\right) \cdot \hat{x}\right)$$

$$\vec{v} \times \nabla \times \vec{A} = \vec{v} \times \left\{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}\right\}$$

$$\equiv \vec{v} \times \left\{\left(\nabla \times \vec{A}\right)_x\hat{x} + \left(\nabla \times \vec{A}\right)_y\hat{y} + \left(\nabla \times \vec{A}\right)_z\hat{z}\right\}$$

하전 입자의 운동 방정식

$$\vec{v} \times \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_s \\ (\nabla \times \vec{A})_x & (\nabla \times \vec{A})_y & (\nabla \times \vec{A})_z \end{vmatrix}$$

$$= \left(v_y(\nabla \times \vec{A})_z - v_z(\nabla \times \vec{A})_y\right)\hat{x} - \left(v_x(\nabla \times \vec{A})_z - v_z(\nabla \times \vec{A})_x\right)\hat{y} + \left(v_x(\nabla \times \vec{A})_y - v_y(\nabla \times \vec{A})_x\right)\hat{z}$$

$$(\vec{v} \times \nabla \times \vec{A}) \cdot \hat{x} = v_y A_3 - v_z A_2$$

$$= v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

$$= v_y \frac{\partial A_y}{\partial x} - v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_z}{\partial x} - v_z \frac{\partial A_x}{\partial z}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt} + \frac{\partial A_x}{\partial t}$$

$$= \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z + \frac{\partial A_x}{\partial t}$$

$$(\vec{v} \times \nabla \times \vec{A}) \cdot \hat{x}$$

$$= v_y \frac{\partial A_y}{\partial x} - v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_z}{\partial x} - v_z \frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial x}v_x - \frac{\partial A_x}{\partial x}v_x$$

$$= v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} + v_x \frac{\partial A_x}{\partial x} - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} = \vec{v} \cdot \frac{\partial \vec{A}}{\partial x} - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

하전 입자의 운동 방정식

$$Q_x = q\left(-\nabla V\cdot\hat{x} - \frac{\partial}{\partial t}\vec{A}\cdot\hat{x} + \left(\vec{v}\times\nabla\times\vec{A}\right)\cdot\hat{x}\right)$$

$$= q\left(-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} + \vec{v}\cdot\frac{\partial\vec{A}}{\partial x} - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}\right)$$

$$= q\left(-\frac{\partial V}{\partial x} + \vec{v}\cdot\frac{\partial\vec{A}}{\partial x} - \frac{dA_x}{dt}\right)$$

$$Q_x = q\left(-\frac{\partial}{\partial x}\left(V - \vec{v}\cdot\vec{A}\right) - \frac{dA_x}{dt}\right)$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{x}} - \frac{\partial T}{\partial x} = -q\frac{\partial}{\partial x}\left(V - \vec{v}\cdot\vec{A}\right) - q\frac{dA_x}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}} + qA_x\right) - \frac{\partial}{\partial x}\left(T - qV + q\vec{v}\cdot\vec{A}\right) = 0$$

$$A_x = \frac{\partial}{\partial\dot{x}}\left(\vec{A}\cdot\vec{v}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}} + q\frac{\partial}{\partial\dot{x}}\left(\vec{A}\cdot\vec{v}\right)\right) - \frac{\partial}{\partial x}\left(T - qV + q\vec{v}\cdot\vec{A}\right) = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial\dot{x}}(\mathcal{L}) - \frac{\partial}{\partial x}(\mathcal{L}) = 0 \qquad \mathcal{L} = T - qV + q\vec{v}\cdot\vec{A}$$

예제 4

From Wikipedia

숙제) 육극 자석 내에서의 운동 방정식을 구하시오.

$$\vec{B} = \left(\lambda xy, \frac{1}{2}\lambda(x^2 - y^2), 0\right) \qquad \lambda = 6\mu_0 \frac{nI}{a^3}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0 \quad \Rightarrow \quad A_x = A_y = 0$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y} = \lambda xy$$

$$B_y = -\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) = -\frac{\partial A_z}{\partial x} = \frac{1}{2}\lambda(x^2 - y^2)$$

$$\therefore \vec{A} = \left(0, 0, -\frac{1}{6}\lambda(x^3 - 3xy^2)\right)$$

예제 4

$$\vec{A} = \left(0, 0, -\frac{\lambda}{6}(x^3 - 3xy^2)\right)$$

$$\mathcal{L} = T - qV + q\vec{v}\cdot\vec{A} = -\frac{mc^2}{\gamma} - qV + q\vec{v}\cdot\vec{A}$$

$$= -mc^2\sqrt{1 - \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{c^2}} - q\frac{\lambda}{6}(x^3 - 3xy^2)\dot{z}$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}(\mathcal{L}) - \frac{\partial}{\partial x}(\mathcal{L}) = 0$$

$$\frac{\partial}{\partial \dot{x}}(\mathcal{L}) = \gamma m\dot{x}$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}(\mathcal{L}) = \dot{\gamma} m\dot{x} + \gamma m\ddot{x}$$

$$\frac{\partial}{\partial x}(\mathcal{L}) = -\frac{q\lambda}{2}(x^2 - y^2)\dot{z}$$

$$\therefore\ \dot{\gamma} m\dot{x} + \gamma m\ddot{x} + \frac{q\lambda}{2}(x^2 - y^2)\dot{z} = 0$$

강의 요약

- 지렛대의 원리

$$F_1 l_1 = F_2 l_2 + F_3 l_3$$

- 가상의 일의 원리

$$\delta W = F \cdot \delta l$$

- 달랑베르의 원리

$$\left(\vec{F} - \frac{d\vec{p}}{dt}\right) \cdot \delta\vec{r} = 0$$

- 라그랑지 방정식

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(\mathcal{L}) - \frac{\partial}{\partial q_j}(\mathcal{L}) = 0$$

- 하전입자의 라그랑지안

$$\mathcal{L} = -\frac{mc^2}{\gamma} - qV + q\vec{v} \cdot \vec{A}$$