2022 ATE accelerator school

입자가속기의 종류와 원리 개론: Part 2

UNIST
Department of Physics
Prof. Moses Chung

포항가속기 연구소, 과학관1층 대강당

2022.08.08~2022.08.12

mchung@unist.ac.kr, https://mchung.unist.ac.kr/

→ Covered by our magnet experts!

Magnetic scalar potential

- Field free ($\mathbf{J} = 0$) vacuum region ($\mu = \mu_0$):

Sometimes (-) sign is omitted for simplicity

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = 0 \quad \longrightarrow \quad \mathbf{B} = -\nabla\psi, \quad \nabla^2\psi = 0$$

- In the limit of a device long compared to its transverse dimensions:

$$\nabla^2\psi \approx \nabla_\perp^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

- The solution of the above equation are of a form that is well behaved on axis (by separation of variables):

$$\psi = \sum_{n=1}^{\infty} a_n\rho^n\cos(n\phi) + b_n\rho^n\sin(n\phi)$$

→ Be careful ! Index convention (n from 1 vs. n from 0) differs in Europe and US, and by authors and textbooks

Multipoles

- For $n = 1$:

$$\psi_1 = a_1 \rho \cos(\phi) + b_1 \rho \sin(\phi) = a_1 x + b_1 y$$

→ Equipotential surfaces form lines

$$\mathbf{B}_1 = -\nabla \psi_1 = -\left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \right) \psi_1 = -a_1 \hat{x} - b_1 \hat{y}$$

Skew dipole

Dipole

- For $n = 2$:

$$\psi_2 = a_2 \rho^2 \cos(2\phi) + b_2 \rho^2 \sin(2\phi) = a_2(x^2 - y^2) + 2b_2 xy$$

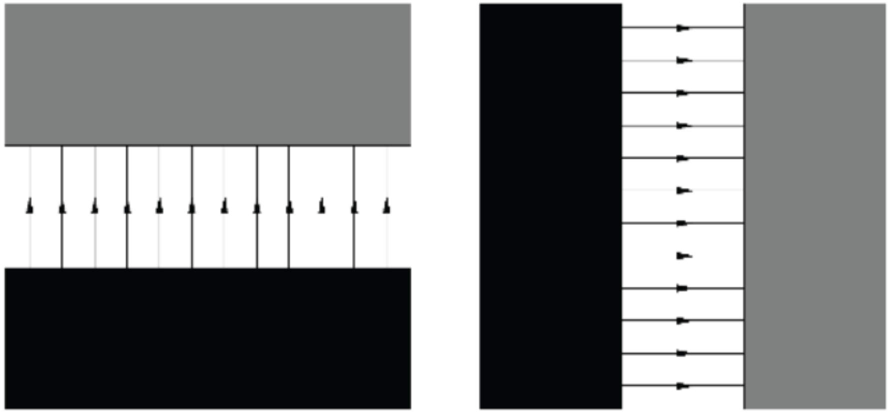
→ Equipotential surfaces form hyperbolae

$$\mathbf{B}_2 = -\nabla \psi_2 = -\left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \right) \psi_2 = 2a_2(-x\hat{x} + y\hat{y}) - 2b_2(y\hat{x} + x\hat{y})$$

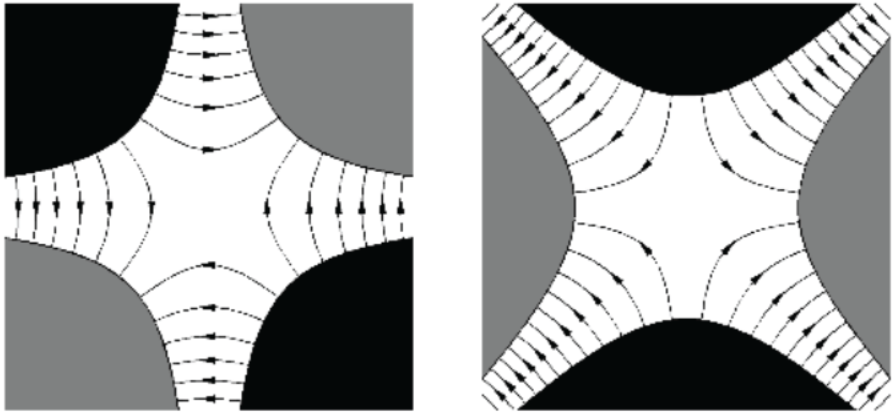
Skew quadrupole

Quadrupole

Dipole and Skew dipole



Quadrupole and Skew quadrupole



Motion in quadrupole fields

- Force due to quadrupole fields:

$$\mathbf{F}_\perp = qv_z\hat{z}\times\mathbf{B}_2 = -2qv_zb_2(y\hat{y}-x\hat{x})$$

- Meaning of the coefficient b_2: Measure of field gradient

$$-2b_2 = \left.\frac{\partial B_x}{\partial y}\right|_{(0,0)} = \left.\frac{\partial B_y}{\partial x}\right|_{(0,0)} \equiv B'$$

- Transverse equations of motion for a momentum p_0, assuming paraxial motion near the z-axis:

$$\begin{aligned}\frac{d^2x}{dz^2} &= x'' = \frac{F_x}{\gamma m_0v_0^2} = \frac{+2qv_zb_2x}{\gamma m_0v_0^2} = -\frac{qB'}{p_0}x\\ \frac{d^2y}{dz^2} &= y'' = \frac{F_y}{\gamma m_0v_0^2} = \frac{-2qv_zb_2y}{\gamma m_0v_0^2} = +\frac{qB'}{p_0}y\end{aligned}$$

- In standard oscillator form:

$$x''+\kappa_0^2x=0,\quad y''-\kappa_0^2y=0$$

- Here, the square wave number is sometimes known as the focusing strength:

$$\kappa_0^2 \equiv \frac{qB'}{p_0} = \frac{B'}{[B\rho]} = K$$

Motion in quadrupole fields (cont'd)

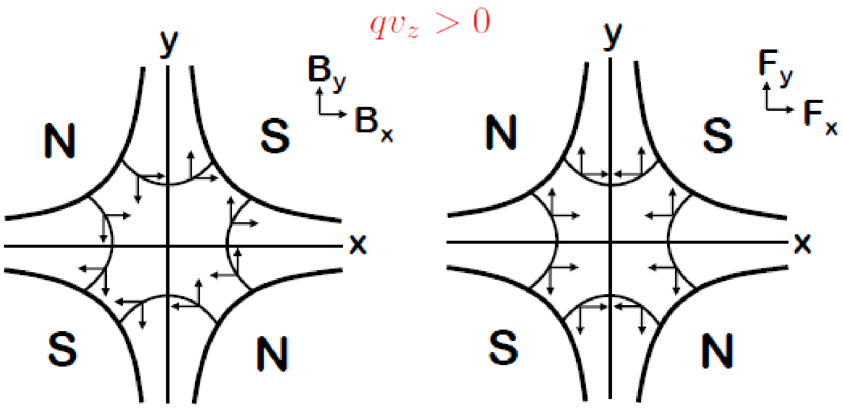
- For $\kappa_0^2 > 0$, one has simple harmonic oscillation in *x* (around *x=0*), and the motion in *y* is hyperbolic.

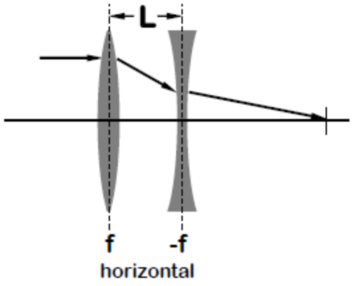
$$x = x_0 \cos\left[\kappa_0(z - z_0)\right] + \frac{x_0'}{\kappa_0}\sin\left[\kappa_0(z - z_0)\right] \qquad \text{with} \quad x(z_0) = x_0, \quad x'(z_0) = x_0'$$

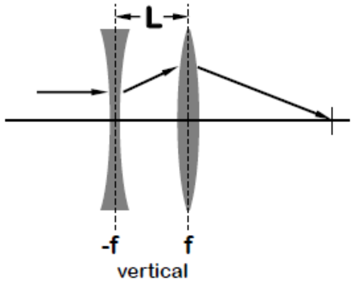
$$y = y_0 \cosh\left[\kappa_0(z - z_0)\right] + \frac{y_0'}{\kappa_0}\sinh\left[\kappa_0(z - z_0)\right] \qquad \text{with} \quad y(z_0) = y_0, \quad y'(z_0) = y_0'$$

- For $\kappa_0^2 < 0$, the motion is simple harmonic(oscillatory) in *y*, and hyperbolic(unbounded) in *x.*

- Focusing with quadrupoles alone can only be accomplished in one transverse direction at a time. Ways of circumventing this apparent limitation in achieving transverse stability, by use of alternating gradient (AG) focusing.

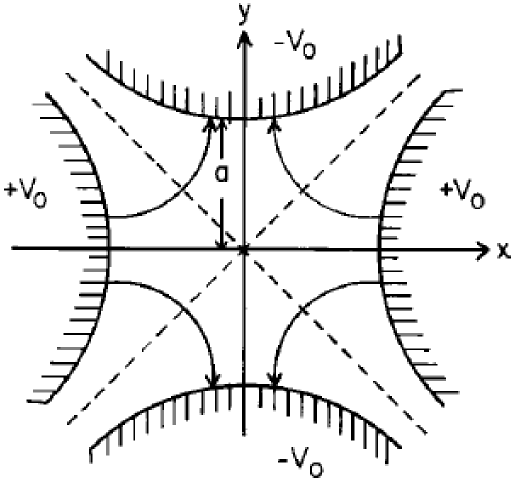




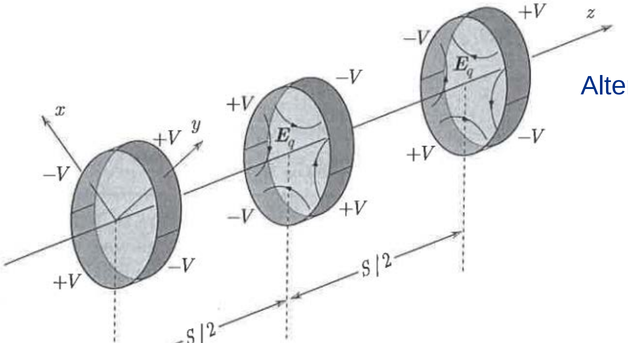


[Note] Electric quadrupole

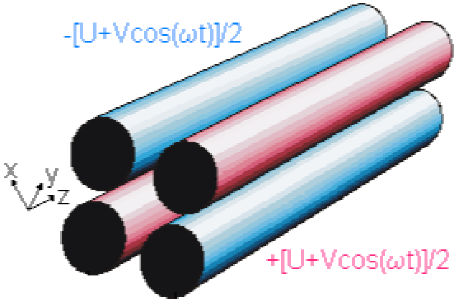
- The commonly encountered level of 1 T static magnetic field is equivalent to a 299.8 MV/m static electric field in force for a relativistic *(v ≈ c)* charged particle.
- This electric field exceeds typical breakdown limits on metallic surfaces by nearly two orders of magnitude, giving partial explanation to the predominance of magnetostatic devices over electrostatic devices for manipulation of charged particle beams.
- Therefore, the transverse electric field quadrupole is found mainly in very low energy applications.



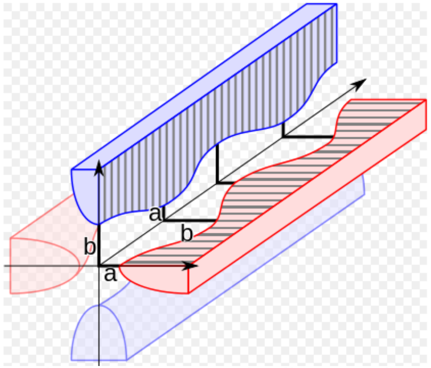
Hyperbolic surfaces rotated by 45 degrees from magnetic case



Alternating in space



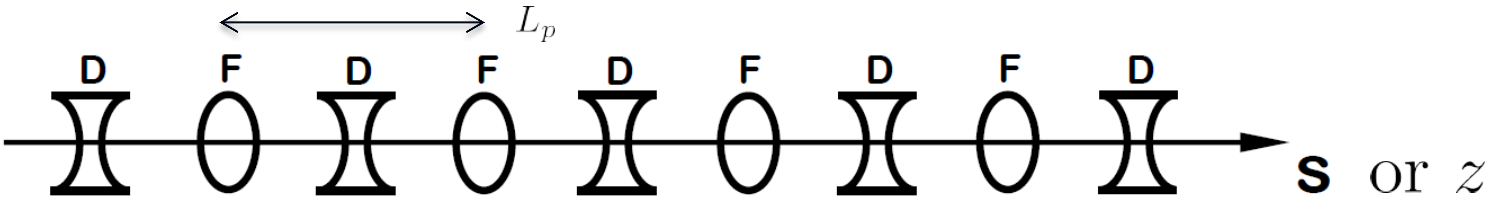
Alternating in time



Alternating in time + longitudinal modulation

Periodic focusing

- Most large accelerators are made up of several (or many) identical modules and, therefore, have periodicity of L_p:
  - Circular machine: $L_p = C/M_p$ → Number of repeated periods along the circumference C
  - Linear machine: array of simple quadrupole magnets with differing sign field gradient



- Hill’s equation:

$$x'' + \kappa_x^2(z)x = 0, \qquad \kappa_x^2(z + L_p) = \kappa_x^2(z) \equiv K_x(z) \text{ in some other books}$$

- Two special cases which can be readily analyzed.
  - The focusing is sinusoidally varying: Mathieu equation
  - The focusing is piece-wise constant : Combination of a number of simple harmonic oscillator solutions

Matrix formalism

- Initial state vector:

$$\mathbf{x}(z_0) = \left(\begin{array}{c} x \\ x' \end{array} \right)_{z=z_0} = \left(\begin{array}{c} x_i \\ x'_i \end{array} \right) = (x \quad x'_i)^T$$

- Solution of the simple harmonic oscillator for $\kappa_0^2 > 0$:

$$\begin{aligned} x(z) &= x_i \cos[\kappa_0(z-z_0)] + \frac{x'_i}{\kappa_0} \sin[\kappa_0(z-z_0)] \\ x'(z) &= -\kappa_0 x_i \sin[\kappa_0(z-z_0)] + x'_i \cos[\kappa_0(z-z_0)] \end{aligned}$$

  - If conveniently expressed by a matrix expression:

$$\mathbf{x}(z) = \mathbf{M}_F \cdot \mathbf{x}(z_0)$$

$$\mathbf{M}_F = \left[\begin{array}{cc} \cos[\kappa_0(z-z_0)] & \frac{1}{\kappa_0}\sin[\kappa_0(z-z_0)] \\ -\kappa_0 \sin[\kappa_0(z-z_0)] & \cos[\kappa_0(z-z_0)] \end{array} \right]$$

  - Through a focusing section of length l:

$$\mathbf{M}_F = \left[\begin{array}{cc} \cos[\kappa_0 l] & \frac{1}{\kappa_0}\sin[\kappa_0 l] \\ -\kappa_0 \sin[\kappa_0 l] & \cos[\kappa_0 l] \end{array} \right]$$

Matrix formalism (cont'd)

- Solution of the simple harmonic oscillator for $\kappa_0^2 = -|\kappa_0|^2 < 0$:

$$
\begin{aligned}
x(z) &= x_i \cosh[|\kappa_0|(z-z_0)] + \frac{x_i'}{|\kappa_0|}\sinh[|\kappa_0|(z-z_0)] \\
x'(z) &= |\kappa_0| x_i \sinh[|\kappa_0|(z-z_0)] + x_i' \cosh[|\kappa_0|(z-z_0)]
\end{aligned}
$$

  - If conveniently expressed by a matrix expression:

$$\mathbf{x}(z) = \mathbf{M}_D \cdot \mathbf{x}(z_0)$$

$$\mathbf{M}_D = \begin{bmatrix} \cosh[|\kappa_0|(z-z_0)] & \frac{1}{|\kappa_0|}\sinh[|\kappa_0|(z-z_0)] \\ |\kappa_0|\sinh[|\kappa_0|(z-z_0)] & \cosh[|\kappa_0|(z-z_0)] \end{bmatrix}$$

- Limiting cases:
  - Force-free drift: $\kappa_0 \to 0$

Length of drift space

$$\mathbf{M}_F = \mathbf{M}_D = \mathbf{M}_O = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & L_d \\ 0 & 1 \end{bmatrix}$$

The position x changes while the angle x' does not

  - Thin-lens limit: $l \to 0 \ \text{ while } \kappa_0^2 l \text{ is kept finite}$

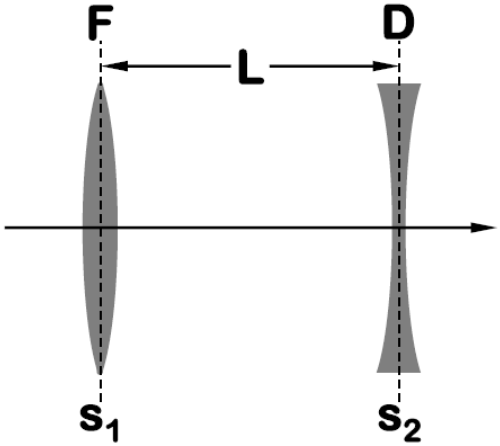
$$\mathbf{M}_{F(D)} = \begin{bmatrix} 1 & 0 \\ \mp\kappa_0^2 l & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{bmatrix}$$

The change in position x is negligible and only the angle x' is transformed

Focal length

[Example 1] Doublet

- Step-by-step matrix multiplication of all individual elements:



$$\mathbf{M}_x^{1\to2} = \begin{bmatrix} 1 & 0 \\ \frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f^*} & 1 + \frac{L}{f_2} \end{bmatrix}$$

$$\frac{1}{f^*} = \frac{1}{f_1} - \frac{1}{f_2} + \frac{L}{f_1 f_2}$$

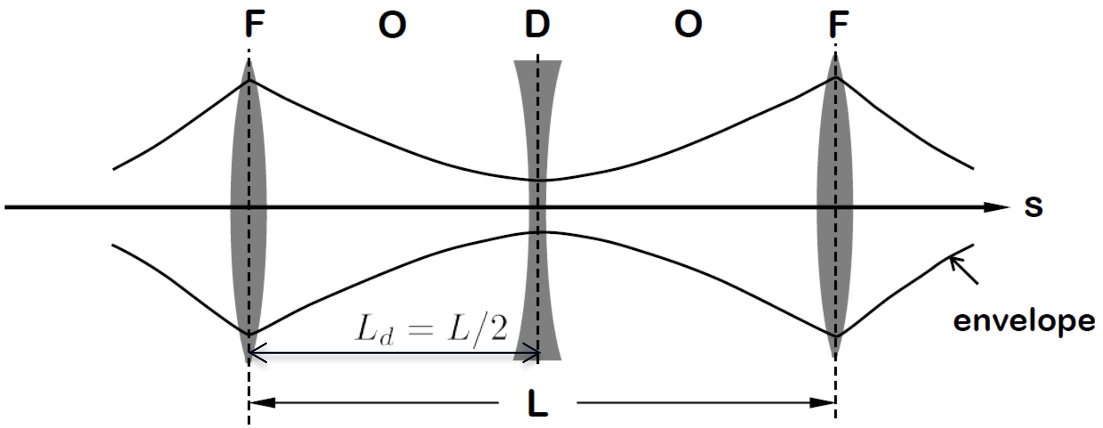
Effective focal length of the system

- For vertical direction: reversing sign of f_1 and f_2

- There is a region of parameters where the sign of f^* is the same and positive for both horizontal and vertical planes (for example, when $f_1 = f_2$), which corresponds to the focusing in both planes.

[Example 2] FODO lattice

- Focus(F)-Drift(O)-Defocus(D)-Drift(O) lattice:



$$\mathbf{x}(z) = \mathbf{x}(L + z_0) = \mathbf{x}(2L_d + 2l + z_0) = \mathbf{M}_O \cdot \mathbf{M}_D \cdot \mathbf{M}_O \cdot \mathbf{M}_F \cdot \mathbf{x}(z_0) = \mathbf{M}_T \cdot \mathbf{x}(z_0)$$

$$\mathbf{M}_T = \begin{bmatrix} 1 - \frac{L_d}{f} - \left(\frac{L_d}{f}\right)^2 & 2L_d + \frac{L_d^2}{f} \\ -\frac{L_d}{f^2} & \frac{L_d}{f} + 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial x_i} & \frac{\partial x}{\partial x'_i} \\ \frac{\partial x'}{\partial x_i} & \frac{\partial x'}{\partial x'_i} \end{bmatrix}$$

What about y direction ?

- Note that the matrix product given above is written in reverse order from that in which the component matrices are physically encountered in the beam line. Confusion on the ordering of matrices is the most common mistake made in the matrix analysis of beam dynamics!

Pseudo-harmonic oscillations

- Let's try for the solution of the Hill's equation in the following form:

A constant determined by initial
conditions of the particle.
In some other books:
$\epsilon \longrightarrow 2J$

A constant determined by initial
conditions of the particle

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\left[\phi(s) - \psi\right]$$

Beta function, proportional
to the square of the
envelope of the oscillation

Phase change of the
oscillation: betatron phase

$$x'(s) = \frac{\beta'(s)}{2}\sqrt{\frac{\epsilon}{\beta(s)}}\cos[\phi(s) - \psi] - \phi'(s)\sqrt{\epsilon\beta(s)}\sin[\phi(s) - \psi]$$

$$x''(s) = \underbrace{\left[\frac{\beta''(s)}{2\sqrt{\beta(s)}} - \frac{\beta(s)'^2}{4\beta(s)^{3/2}} - \sqrt{\beta(s)}\phi'^2(s)\right]}_{= -k(s)\sqrt{\beta(s)}}\sqrt{\epsilon}\cos[\phi(s) - \psi] - \underbrace{\left[\phi''(s)\sqrt{\beta(s)} + \frac{\beta'(s)\phi'(s)}{\sqrt{\beta(s)}}\right]}_{= 0}\sqrt{\epsilon}\sin[\phi(s) - \psi]$$

- New differential equations (depending only on the magnetic lattice)

$$\frac{1}{2}\beta(s)\beta''(s) - \frac{1}{4}\beta'^2(s) + k(s)\beta^2(s) = 1 \qquad \phi'(s) = \frac{1}{\beta(s)}$$

Envelope equation

Phase advance equation

PAL 포항가속기연구소 POHANG ACCELERATOR LABORATORY

Pseudo-harmonic oscillations

UNIST ULSAN NATIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY

- By defining alpha function as $\alpha(s) = -\frac{\beta'(s)}{2}$

Meaning of the alpha function:
slope of the change in the envelope
($\alpha > 0$: converging, $\alpha < 0$: diverging)

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\left[\phi(s) - \psi\right] \qquad x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\{\sin[\phi(s)-\psi] + \alpha(s)\cos[\phi(s)-\psi]\}$$

- With the following initial conditions:

$$\beta(s=s_0) = \beta_0, \quad \alpha(s=s_0) = \alpha_0, \quad \phi(s=s_0) = 0$$

$$x(s=s_0) = x_0 = \sqrt{\epsilon\beta_0}\cos\left[-\psi\right] \qquad x'(s=s_0) = x'_0 = -\sqrt{\frac{\epsilon}{\beta_0}}\{\sin[-\psi] + \alpha_0\cos[-\psi]\}$$

$$\longrightarrow \sqrt{\epsilon}\cos\psi = \frac{x_0}{\sqrt{\beta_0}}, \quad \sqrt{\epsilon}\sin\psi = \alpha_0\frac{x_0}{\sqrt{\beta_0}} + \beta_0 x'_0$$

- Using trigonometric identities:

$$\begin{aligned} x(s) &= \sqrt{\epsilon\beta(s)}\cos\left[\phi(s)-\psi\right] = \sqrt{\epsilon\beta(s)}\left[\cos\phi(s)\cos\psi + \sin\phi(s)\sin\psi\right] \\ &= x_0\left[\sqrt{\frac{\beta(s)}{\beta_0}}\{\cos\phi(s) + \alpha_0\sin\phi(s)\}\right] + x'_0\left[\sqrt{\beta(s)\beta_0}\sin\phi(s)\right] \\ &\equiv x_0 C(s) + x'_0 S(s) \end{aligned}$$

Connection with matrix formalism

- The elements of the transfer matrix can be expressed via the Twiss functions (α, β, γ) at the beginning and end of the beam line:

$$\begin{aligned} x(s) &= x_0 C(s) + x_0' S(s) \\ x'(s) &= x_0 C'(s) + x_0' S'(s) \end{aligned}$$

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$\mathbf{M}_{s_0 \to s} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\beta(s)}{\beta_0}} \{\cos\Delta\phi + \alpha_0 \sin\Delta\phi\} & \sqrt{\beta(s)\beta_0} \sin\Delta\phi \\ -\frac{(\alpha(s)-\alpha_0)\cos\Delta\phi + (1+\alpha(s)\alpha_0)\sin\Delta\phi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\frac{\beta(s)}{\beta_0}} \{\cos\Delta\phi - \alpha(s)\sin\Delta\phi\} \end{bmatrix}$$

where

$$\Delta\phi = \phi(s) - \underbrace{\phi(s_0)}_{=0} = \phi(s) = \int_{s_0}^{s} \frac{ds'}{\beta(s')}$$

- One can also have the following decomposition:

$$\begin{aligned} \mathbf{M}_{s_0 \to s} &= \begin{bmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{bmatrix} \times \begin{bmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{bmatrix} \\ &= \mathbf{B}(s) \begin{bmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{bmatrix} \mathbf{B}^{-1}(s_0) \end{aligned}$$

CW rotation

Connection with matrix formalism

- So far, we haven't yet assumed any periodicity in the transfer line. However, we may consider a periodic machine, and then the transfer matrix over a single turn (or single lattice period) would reduce to

$$\begin{aligned}\mathbf{M}_{s_0\rightarrow s_0+L_p} &= \begin{bmatrix} \cos\Delta\phi+\alpha_0\sin\Delta\phi & \beta_0\sin\Delta\phi \\ -\frac{(1+\alpha_0^2)}{\beta_0}\sin\Delta\phi & \cos\mu-\alpha_0\sin\Delta\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\mu+\alpha_0\sin\mu & \beta_0\sin\mu \\ -\gamma_0\sin\mu & \cos\mu-\alpha_0\sin\mu \end{bmatrix}\end{aligned}$$

When we impose periodic boundary condition on the beta function

$$\beta(s_0+L_p)=\beta_0$$

where we define gamma function

$$\gamma_0=\frac{1+\alpha_0^2}{\beta_0}$$

and phase advance for one turn (or one period)

$$\mu=\Delta\phi$$

Courant-Snyder invariant

- Hill's equation have a remarkable property: they have an invariant!

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\left[\phi(s) - \psi\right] \qquad x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\left\{\sin[\phi(s)-\psi] + \alpha(s)\cos[\phi(s)-\psi]\right\}$$

$$\longrightarrow \sqrt{\epsilon}\cos\left[\phi(s)-\psi\right] = \frac{x(s)}{\sqrt{\beta(s)}}, \qquad \sqrt{\epsilon}\sin\left[\phi(s)-\psi\right] = \frac{\alpha(s)x(s)}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}x'(s)$$

- Using trigonometric identities:

$$\left(\frac{x(s)}{\sqrt{\beta(s)}}\right)^2 + \left(\frac{\alpha(s)x(s)}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}x'(s)\right)^2 = \epsilon = const.$$

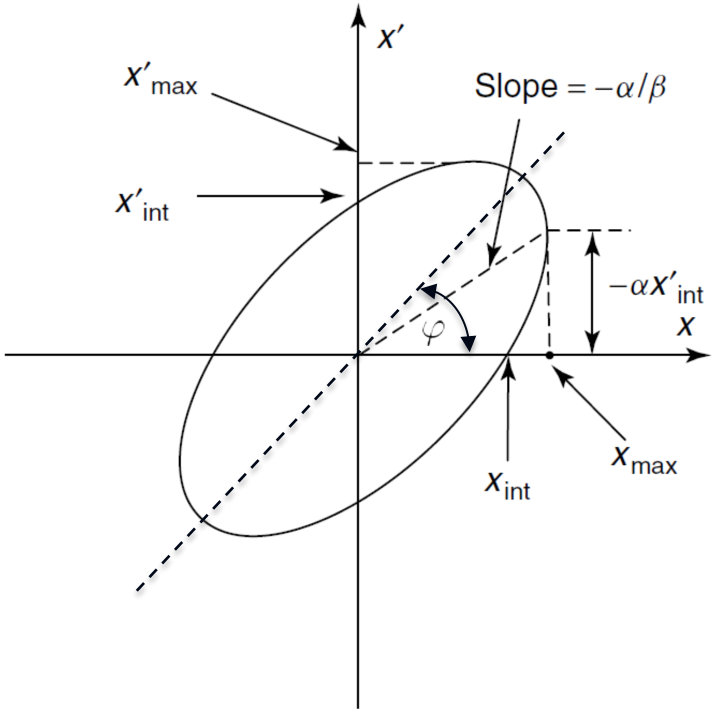
$$\epsilon = \beta(s)x'^2(s) + 2\alpha(s)x(s)x'(s) + \gamma(s)x^2(s) = \beta(s_0)x'^2(s_0) + 2\alpha(s_0)x(s_0)x'(s_0) + \gamma(s_0)x^2(s_0)$$

This invariant is known as Courant-Snyder invariant: Even though an initial point in the trace space $(x(s_0), x'(s_0),)$ changes to a different position $(x(s), x'(s),)$, the Twiss parameters (α, β, γ) change at the same time in such as way that ϵ remains constant.

Phase space (or trace space) ellipse

- The Courant-Snyder invariant defines an (tilted) ellipse in phase space (x, x'):

$$\epsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) = \left(\frac{x(s)}{\sqrt{\beta(s)}}\right)^2 + \left(\frac{\alpha(s)x(s)}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}x'(s)\right)^2$$



$$\tan 2\varphi = \frac{2\alpha}{\gamma - \beta}$$

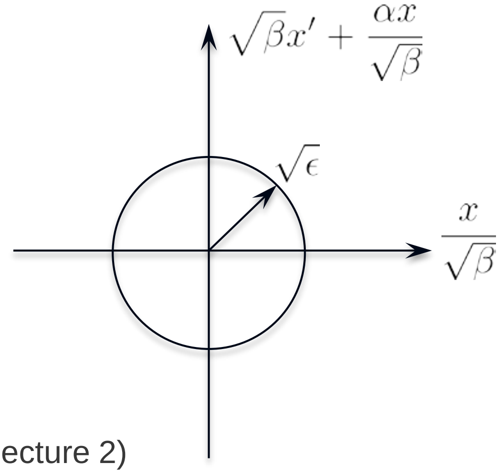
$$\text{Area in phase-space} = \pi\epsilon = const.$$

$[\epsilon]$ = m-rad, or mm-mrad, or π mm-mrad

$$x_{max} = \sqrt{\epsilon\beta}, \quad x_{int} = \sqrt{\epsilon/\gamma}$$
$$x'_{max} = \sqrt{\epsilon\gamma}, \quad x'_{int} = \sqrt{\epsilon/\beta}$$

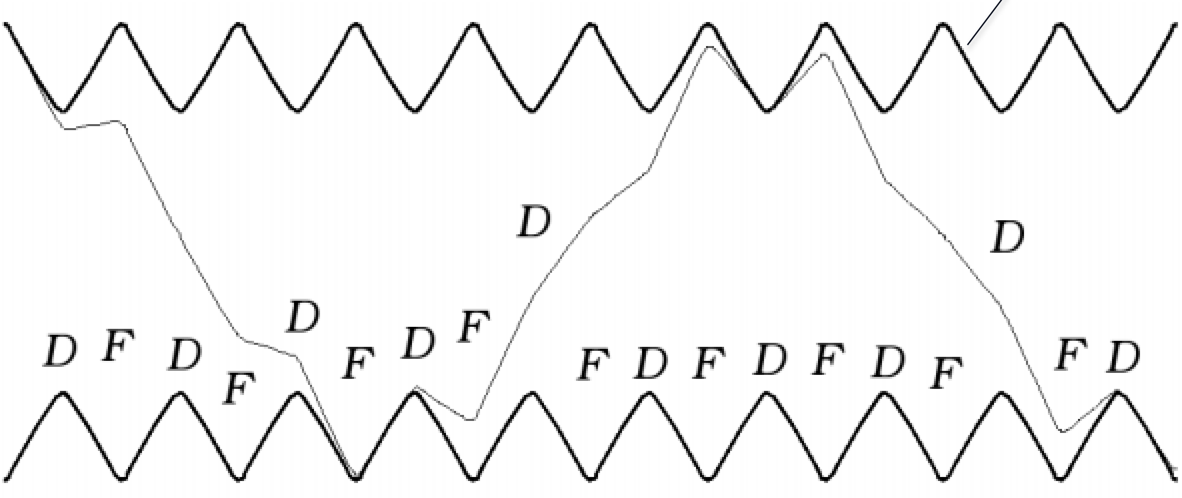
- Or, in the normalized coordinates, it defines a circle:

$$\epsilon = \left(\frac{x(s)}{\sqrt{\beta(s)}}\right)^2 + \left(\frac{\alpha(s)x(s)}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}x'(s)\right)^2 = x_n^2 + x_n'^2$$



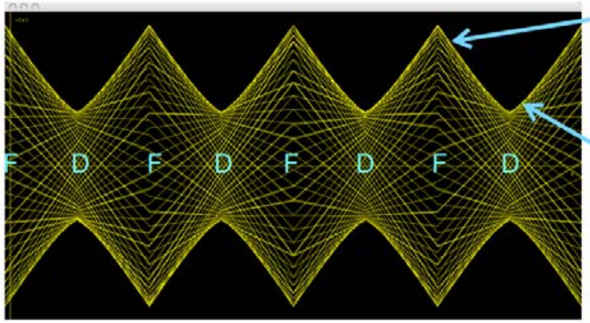
Typical trajectory

- Slow simple harmonic oscillator-like behavior (secular motion) + Fast oscillatory motion with lattice period:



Maximum envelope a particle with arbitrary initial conditions can have

For multi-turns (or multi-particles):



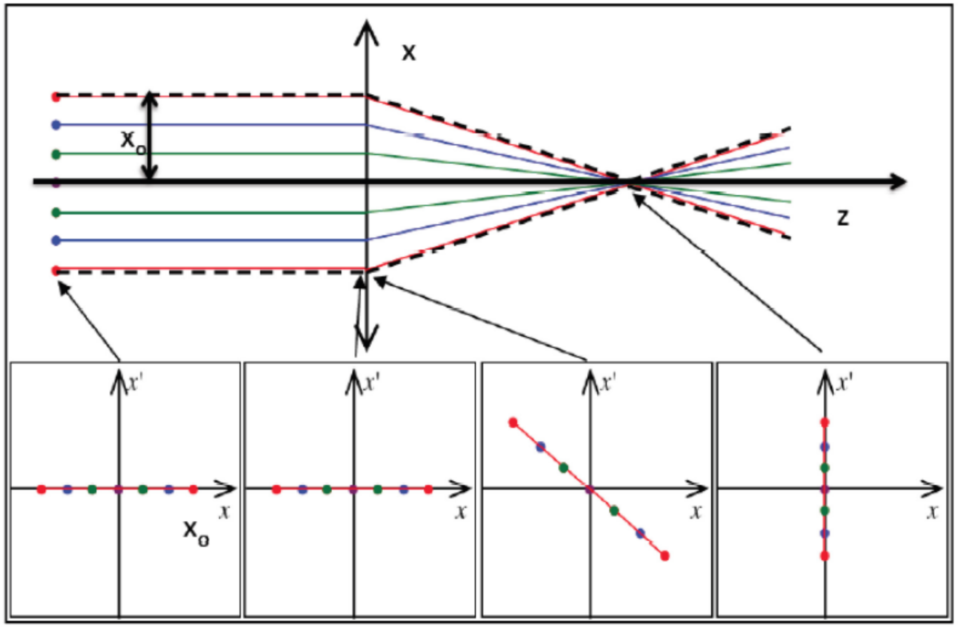
Higher β --
smaller phase advance
larger beam size

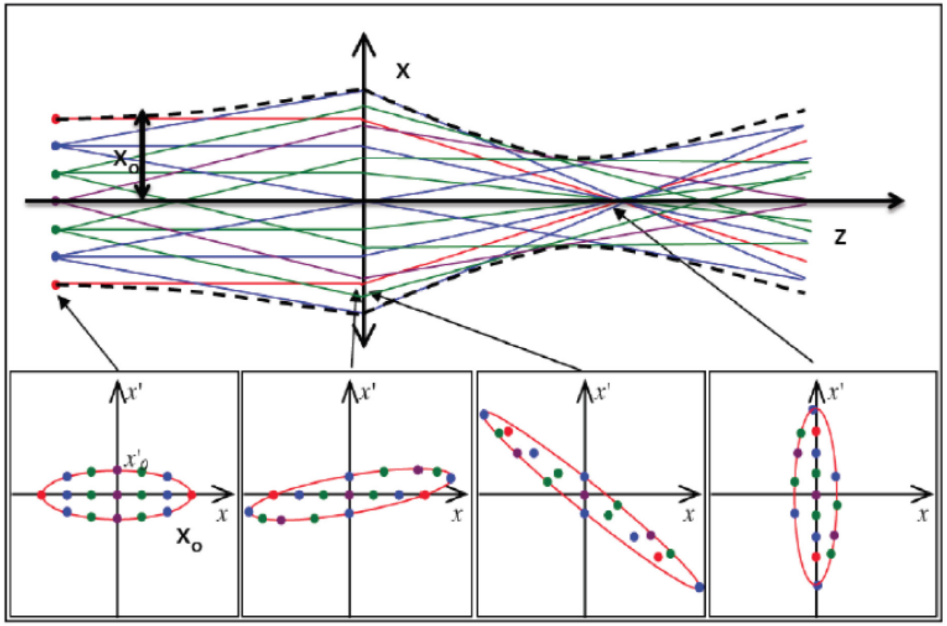
Lower β --
greater phase advance
smaller beam size

Collection of particles: Beam

Single particle

Laminar vs Non-laminar Beams



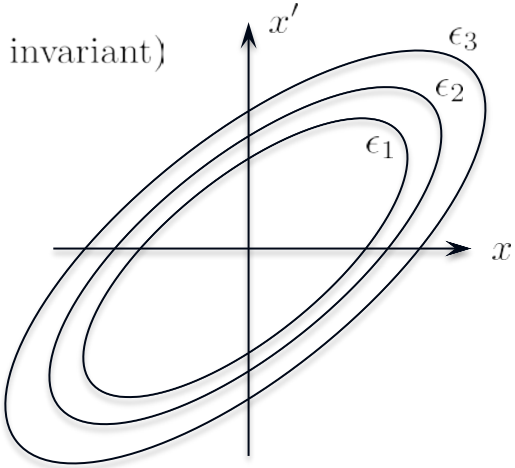


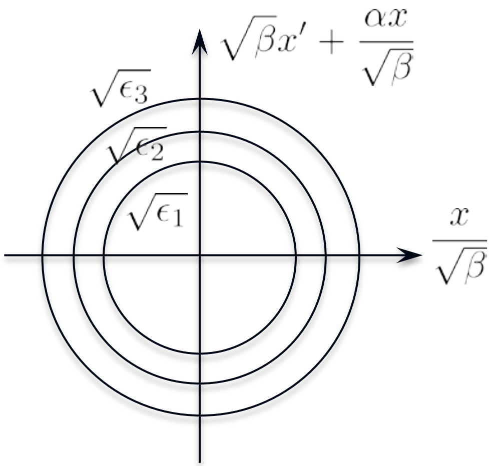
Bi-Gaussian distribution

- We assume the particle distribution is a bi-Gaussian distribution in the following form:

$$f(x,x') = \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\epsilon_{\rm rms}}\right] \propto \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] \propto \exp\left[-\frac{(x/\sqrt{\beta})^2 + (\sqrt{\beta}x' + \alpha x/\sqrt{\beta})^2}{2\epsilon_{\rm rms}}\right]$$

$$\frac{df}{ds} = 0 \ (\epsilon = \text{invariant})$$





Constant (single particle) emittance ellipses define contours of constant phase-space distribution density

Constant (single particle) emittance circles in the normalized coordinates define contours of constant phase-space distribution density

- The rms beam emittance is proportional to the average of all the single particle emittances.
- The rms beam emittance is defined through the ellipse of the exp[-1/2] contour relative to the peak density contour.

Normalization of the distribution function

- First, check the normalization:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,x')dxdx' &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\epsilon_{\mathrm{rms}}}\exp\left[-\frac{\gamma x^2+2\alpha x x'+\beta x'^2}{2\epsilon_{\mathrm{rms}}}\right]dxdx' \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\epsilon_{\mathrm{rms}}}\exp\left[-\frac{x_n^2+x_n'^2}{2\epsilon_{\mathrm{rms}}}\right]dx_n dx_n' \\ &= \int_{0}^{\infty}\frac{1}{2\pi\epsilon_{\mathrm{rms}}}\exp\left[-\frac{\epsilon}{2\epsilon_{\mathrm{rms}}}\right]\pi d\epsilon \\ &= 1\end{aligned}$$

$$x_n = \frac{x}{\sqrt{\beta}}$$

$$x_n' = \sqrt{\beta}x' + \frac{\alpha x}{\sqrt{\beta}}$$

$$\epsilon = x_n^2 + x_n'^2$$

- Meaning of the rms beam emittance:

Integration by parts

$$\begin{aligned}\langle\epsilon\rangle &= \int_0^{\infty}\epsilon\frac{1}{2\epsilon_{\mathrm{rms}}}\exp\left[-\frac{\epsilon}{2\epsilon_{\mathrm{rms}}}\right]d\epsilon \\ &= \frac{1}{2\epsilon_{\mathrm{rms}}}\left\{\epsilon(-2\epsilon_{\mathrm{rms}})\exp\left[-\frac{\epsilon}{2\epsilon_{\mathrm{rms}}}\right]\Big|_0^{\infty} + \int_0^{\infty}2\epsilon_{\mathrm{rms}}\exp\left[-\frac{\epsilon}{2\epsilon_{\mathrm{rms}}}\right]d\epsilon\right\} \\ &= 2\epsilon_{\mathrm{rms}} = 2\langle J\rangle\end{aligned}$$

Action

Moments of the distribution function

- From the general properties of the bi-Gaussian distribution in (x, y) plane:

https://en.wikipedia.org/wiki/Multivariate_normal_distribution

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{\delta x^2}{\sigma_x^2} - 2\rho\frac{\delta x\delta_y}{\sigma_x\sigma_y} + \frac{\delta y^2}{\sigma_y^2}\right)\right]$$

Where

$$\delta x = x - \langle x \rangle\,, \quad \delta y = y - \langle y \rangle$$

$$\sigma_x^2 = \left\langle \delta x^2 \right\rangle, \quad \sigma_y^2 = \left\langle \delta y^2 \right\rangle, \quad \sigma_{xy} = \langle \delta x \delta y \rangle \equiv \rho\sigma_x\sigma_y$$

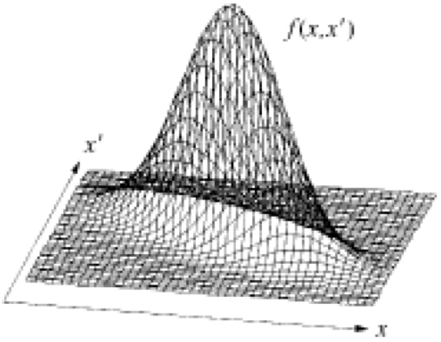
covariance

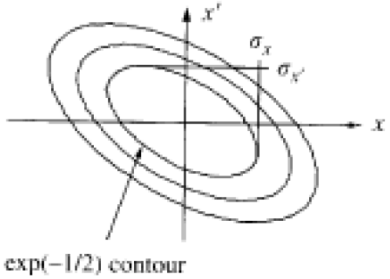
- By comparing with the beam distribution in (x, x') space:

$$\langle x \rangle = \langle x' \rangle = 0 \quad \text{when beam is alinged to its design axis}$$

$$\sigma_x^2 = \left\langle x^2 \right\rangle = \epsilon_{\text{rms}}\beta, \quad \sigma_{x'}^2 = \left\langle x'^2 \right\rangle = \epsilon_{\text{rms}}\gamma, \quad \sigma_{xx'} = \langle xx' \rangle = -\epsilon_{\text{rms}}\alpha$$

$$\epsilon_{\text{rms}} = \sigma_x\sigma_{x'}(1-\rho^2)^{1/2} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \rho^2\sigma_x^2\sigma_{x'}^2} = \sqrt{\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \langle xx' \rangle^2}$$





$\pi\epsilon_{\text{rms}}$ = Area of the exp[-1/2] contour

Beam matrix

- The beam matrix is the second-order moments of the beam distribution:

Contains all the necessary information describing the beam

$$\begin{aligned}\sigma(s)=\mathbf{\Sigma}(s) &= \left\langle \mathbf{x}\mathbf{x}^T \right\rangle \\ &= \begin{bmatrix} \langle x^2\rangle & \langle xx'\rangle \\ \langle xx'\rangle & \langle x'^2\rangle \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_x'^2 \end{bmatrix} \\ &= \underbrace{\epsilon_{\mathrm{rms}}}_{\text{Beam property}} \underbrace{\begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}}_{\text{Lattice properties}} \end{aligned}$$

If the beam aligns with Courant-Snyder parameters

- Note that the determinant of the beam matrix is the rms emittance:

$$\det(\sigma) = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = \epsilon_{\mathrm{rms}}^2$$

- If the transfer matrix is known,

$$\mathbf{x}(s) = \mathbf{M}_{s_0\to s}\cdot\mathbf{x}(s_0)$$

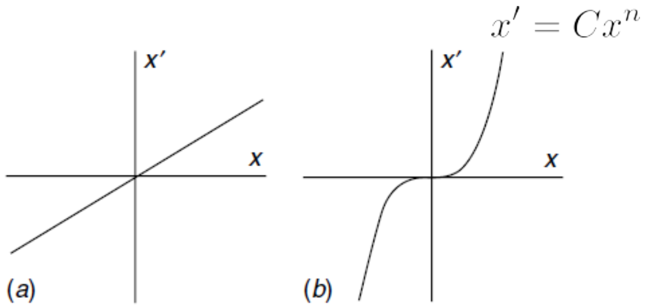
$$\begin{aligned}\sigma(s) &= \left\langle \mathbf{x}(s)\mathbf{x}^T(s)\right\rangle \\ &= \left\langle \mathbf{M}_{s_0\to s}\cdot\mathbf{x}(s_0)\mathbf{x}^T(s_0)\cdot\mathbf{M}^T_{s_0\to s}\right\rangle \\ &= \mathbf{M}_{s_0\to s}\cdot\sigma(s_0)\cdot\mathbf{M}^T_{s_0\to s}\end{aligned}$$

RMS Emittance

- In the case of a real beam with a finite number of particles (N), an RMS emittance can be defined for an effective phase-space (or trace-space) area (or volume).

$$\epsilon_{\rm rms} = \sqrt{\langle x^2\rangle \langle p_x^2\rangle - \langle xp_x\rangle^2}, \text{ or } \sqrt{\langle x^2\rangle \langle x'^2\rangle - \langle xx'\rangle^2}$$

→Depends not only on the true area occupied by the beam in phase space, but also on the distortions produced by nonlinear forces.

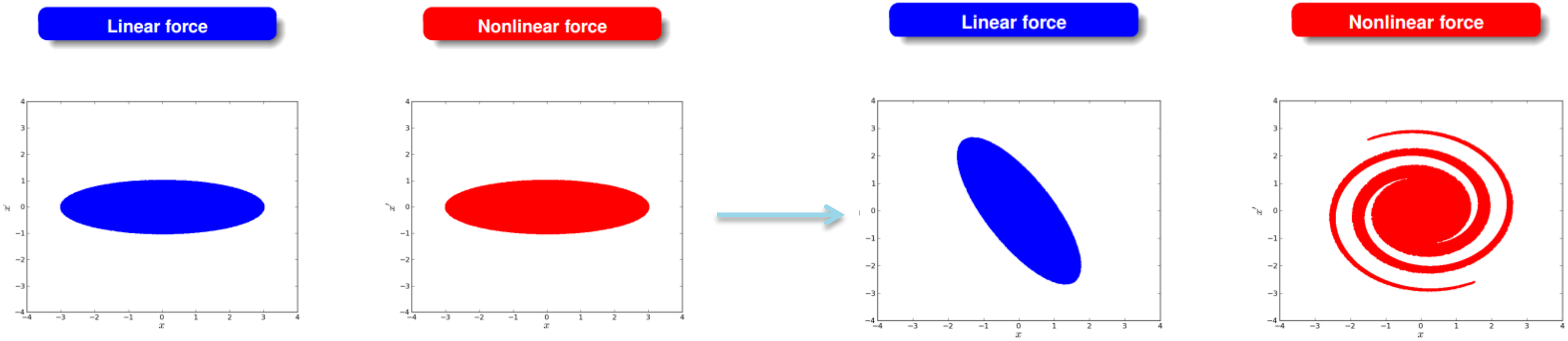


Phase-space area = 0
RMS emittance = 0

Phase-space area = 0
RMS emittance > 0

- However, when nonlinear forces act on the system, e.g. nonlinear magnetic fields, space charge force, the RMS emittance is not conserved.

Filamentation
→ Dilution of phase space density



Normalized emittance

- In the paraxial approximation,

Decrease when there is an acceleration

$$x' = \frac{dx}{ds} = \frac{v_x}{v_z} \simeq \frac{p_x}{p_0}, \quad p_0 = \beta_0 \gamma_0 m_0 c$$

Referece momentum

- We introduced the normalized emittance:

$$\epsilon_n = \beta_0 \gamma_0 \epsilon_{\text{rms}}$$

$$\epsilon_n^2 = (\beta_0 \gamma_0)^2 \left[\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \right] = (m_0 c)^{-2} \left[\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 \right]$$

constant

- The normalized emittance (not the rms emittance in trace space) is, in fact, invariant under combined effects of linear transverse forces and longitudinal acceleration.

- This result is a direct consequence of the adibatic damping of beam particle angle under acceleration, which causes the emittance defined in trace space to be diminished.

- The invariant normalized emittance is an effective area occupied by the beam in the phase plane, not the trace plane.

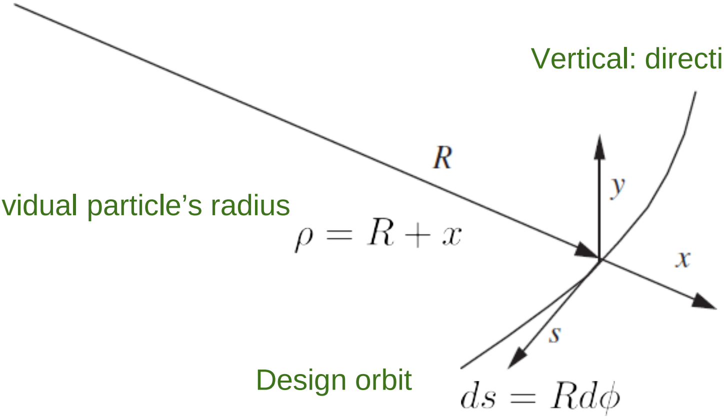
Linear

Circular

Cicular acclerator

- We analyze the charged particle dynamics near the design orbit. The design orbit is specified by a certain radius of curvature (R) and a certain momentum ($p_0 = qB_0R$)

- A new locally defined right-handed coordinate system:



Vertical: direction of dipole magnet field

Individual particle's radius

Horizontal: deviation away from the design orbit

Design orbit

- Equation of motion in this new coordinate system:

$$\frac{dp_\rho}{dt} = \frac{\gamma m_0 v_\phi^2}{\rho} - q v_\phi B_0 \tag{2.9}$$

- The azimuthal velocity and radial momentum:

Reference

$$v_\phi = \rho\dot{\phi} \neq \dot{s} \equiv v_0, \quad \text{and} \quad p_\rho = \gamma m_0 \dot{\rho} = \gamma m_0 \dot{x} = p_x$$

$$v_\phi \approx v_0$$

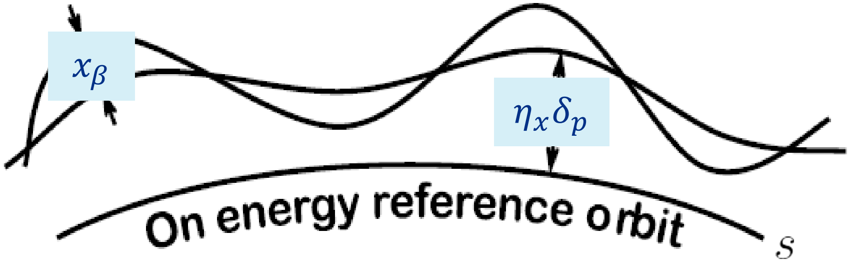
Individual particle's velocity

Dispersion (η or D)

- Change in the design orbit for the off-momentum particle:

$$x = x_\beta + \eta_x \frac{(p - p_0)}{p_0} = x_\beta + \eta_x \frac{\Delta p}{p_0} = x_\beta + \eta_x \delta_p$$

Offset in position   Offset in momentum



- Lowest order Taylor series expansion about the design orbit equilibrium ($p_x = p_\rho = 0$ at $\rho = R$):

$$\frac{dp_x}{dt} = \frac{\gamma m_0 v_0^2}{R_0(1 + x/R_0)} - q v_0 B_0 \simeq \frac{\gamma m_0 v_0^2}{R_0}(1 - x/R_0 + \cdots) - q v_0 B_0$$

$$\simeq -\frac{\gamma m_0 v_0^2}{R_0^2} x + \cancel{\frac{\gamma m_0 v_0^2}{R_0}} - \cancel{q v_0 B_0}$$

$$B_0 R_0 = \frac{p_0}{q}$$

Now we allow v to be deviated from v_0

$$\frac{dp_x}{dt} = \frac{\gamma m_0 v^2}{R_0(1 + x/R_0)} - q v B_0 \simeq \frac{\gamma m_0 v^2}{R_0}(1 - x/R_0 + \cdots) - q v B_0$$

$$\simeq -\frac{\gamma m_0 v^2}{R_0^2} x + \frac{\gamma m_0 v^2}{R_0} - q v B_0$$

$$\simeq -\frac{\gamma m_0 v^2}{R_0^2} x + \gamma m_0 v^2 \left[\frac{1}{R_0} - \frac{q B_0}{p} \right]$$

Path length focusing term   New term caused by $p \neq p_0$

Governing equation for dispersion

- We can express the new force balance equation using s as an independent variable:

$$(\cdots)' \equiv \frac{d}{ds} = \frac{d}{vdt}, \quad p_x = \gamma m_0 \frac{dx}{dt}$$

$$x'' = -\frac{1}{R_0^2}x + \left[\frac{1}{R_0} - \frac{qB_0}{p}\right] \simeq -\frac{1}{R_0^2}x + \left[\frac{1}{R_0} - \frac{1}{R_0}\left(1 - \frac{\Delta p}{p_0}\right)\right] = -\frac{1}{R_0^2}x + \frac{1}{R_0}\frac{\Delta p}{p_0}$$

$$\frac{qB_0}{p} = \frac{1}{R_0}\frac{p_0}{p} = \frac{1}{R_0}\frac{p_0}{p_0 + \Delta p} \simeq \frac{1}{R_0}\left(1 - \frac{\Delta p}{p_0}\right)$$

- With the quadrupole term included,

$$x'' + \left[\frac{1}{R_0^2} + \frac{qB'}{p_0}\right]x = \frac{1}{R_0}\frac{\Delta p}{p_0}$$

1st order in position offset

1st order in momentum offset

- If we substitute $x = x_\beta + \eta_x \frac{\Delta p}{p_0}$

$$x_\beta'' + \left[\frac{1}{R_0^2} + \frac{qB'}{p_0}\right]x_\beta = 0, \quad \eta_x'' \frac{\Delta p}{p_0} + \left[\frac{1}{R_0^2} + \frac{qB'}{p_0}\right]\eta_x \frac{\Delta p}{p_0} = \frac{1}{R_0}\frac{\Delta p}{p_0}$$

$$\eta_x'' + \underbrace{\left[\frac{1}{R_0^2} + \frac{qB'}{p_0}\right]}_{\equiv \kappa_b^2}\eta_x = \frac{1}{R_0}$$

Solution of the dispersion equation

- For net horizontal focusing, the general solution is composed of homogeneous and particular solutions:

$$\eta_x = \underbrace{A\cos(\kappa_b s) + B\sin(\kappa_b s)}_{\text{homogeneous}} + \underbrace{\frac{1}{\kappa_b^2 R_0}}_{\text{particular}}$$

[Note] If there is only bending magnet (i.e., $B' = 0$, no quadrupole),

$$\eta_{x,part} = \frac{1}{\kappa_b^2 R_0} = R_0$$

- If we apply matching boundary conditions at the entrance of the bend magnet ($s = 0$),

$$\eta_x(s) = \left[\eta_x(0) - \frac{1}{\kappa_b^2 R_0}\right]\cos(\kappa_b s) + \frac{\eta_x'(0)}{\kappa_b}\sin(\kappa_b s) + \frac{1}{\kappa_b^2 R_0}$$

$$\eta_x'(s) = \left[\frac{1}{\kappa_b R_0} - \kappa_b \eta_x(0)\right]\sin(\kappa_b s) + \eta_x'(0)\cos(\kappa_b s)$$

Transfer matrix of dispersion

- In the matrix form,

$$\begin{bmatrix} \eta_x(s) \\ \eta_x'(s) \\ 1 \end{bmatrix} = \begin{bmatrix} \cos[\kappa_b s] & \frac{1}{\kappa_b}\sin[\kappa_b s] & \frac{1-\cos(\kappa_b s)}{\kappa_b^2 R_0} \\ -\kappa_b \sin[\kappa_b s] & \cos[\kappa_b s] & \frac{\sin(\kappa_b s)}{\kappa_b R_0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_x(0) \\ \eta_x'(0) \\ 1 \end{bmatrix}$$

[Note]

1. Even if there is no dispersion in the beginning (i.e., $\eta_x(0) = \eta_x'(0) = 0$), dispersion can be created when the beam is transported through a bending magnet.
2. In a straight section ($R_0 \to \infty$, i.e., no bending),

$$\begin{bmatrix} \eta_x(s) \\ \eta_x'(s) \\ 1 \end{bmatrix} = \begin{bmatrix} \cos[\kappa_b s] & \frac{1}{\kappa_b}\sin[\kappa_b s] & 0 \\ -\kappa_b \sin[\kappa_b s] & \cos[\kappa_b s] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_x(0) \\ \eta_x'(0) \\ 1 \end{bmatrix}$$

3. Even in the straight section, dispersion can exist if there is dispersion in the beginning (i.e., $\eta_x(0) \neq 0, \eta_x'(0) \neq 0$).

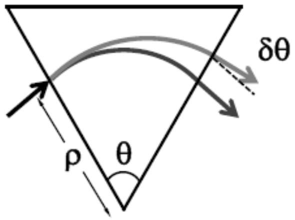


FIGURE 2.18 Bending magnet creates dispersion.

Longitudinal coordinate

- The canonical dependent coordinate in the longitudinal direction is time of arrival relative to the design particle.

$$\tau = t - t_0 \quad \begin{cases} \text{Early particle (head): } < 0 \\ \text{Late particle (tail): } > 0 \end{cases}$$

- In the Hamiltonian analysis, it is useful to introduce a parametrization of the time through a spatial variable,

$$\zeta = -v_0\tau = v_0 t_0 - v_0 t = s - v_0 t = s - \beta_0 ct \quad \begin{cases} \text{Early particle (head): } > 0 \\ \text{Late particle (tail): } < 0 \end{cases}$$

[Note] This is the distance that must be traveled at the design velocity by the design particle, to reach the position of the temporally advanced (or delayed) particle.

[Note] In some books or codes (such as Wolski's book or MAD), the following notations are used.

$$z = \frac{s}{\beta_0} - ct$$

$$\delta = \frac{1}{\beta_0}\frac{\Delta E}{E_0} \simeq \beta_0\delta_p$$

Momentum compaction

- The time of flight of an off-momentum particle through travel distance $L(p)$:

$$t(p) = \frac{L(p)}{v(p)}$$

- First order expansion with paraxial approximation yields

$$\delta t = \delta \tau = \frac{\delta L}{v_z} - \frac{L}{v_z^2}\delta v_z \simeq \frac{\delta L}{v_0} - \frac{L_0}{v_0^2}\delta v_z$$

$$\frac{\delta \tau}{t_0} = \frac{\delta L}{L_0} - \frac{\delta v_z}{v_0} \simeq \left[\alpha_c - \frac{1}{\gamma_0^2}\right]\frac{\delta p}{p_0}$$

$$t_0 = \frac{L_0}{v_0}$$

Here we define the path length parameter (usually called, momentum compaction) as

$$\alpha_c \equiv \frac{\delta L/L_0}{\delta p/p_0}$$

which characterizes the path length changes according to the momentum offset. We also used

$$\frac{\delta v_z}{v_0} \simeq \frac{\delta \beta}{\beta_0} \simeq \frac{1}{\gamma_0^2}\frac{\delta p}{p_0}$$

$$\begin{aligned} \delta p &= \delta(mc\gamma\beta) = mc(\beta\delta\gamma + \gamma\delta\beta) \\ \delta\gamma &= \gamma^3\beta\delta\beta \end{aligned}$$

Phase slip factor (or time dispersion)

- We define so-called phase slip factor:

$$\frac{\delta\tau}{t_0} \simeq \left[\alpha_c - \frac{1}{\gamma_0^2}\right]\frac{\delta p}{p_0} \equiv \eta_\tau \frac{\delta p}{p_0}$$

$$\eta_\tau \equiv \frac{\partial(\delta\tau/t_0)}{\partial(\delta p/p_0)} = \alpha_c - \frac{1}{\gamma_0^2}$$

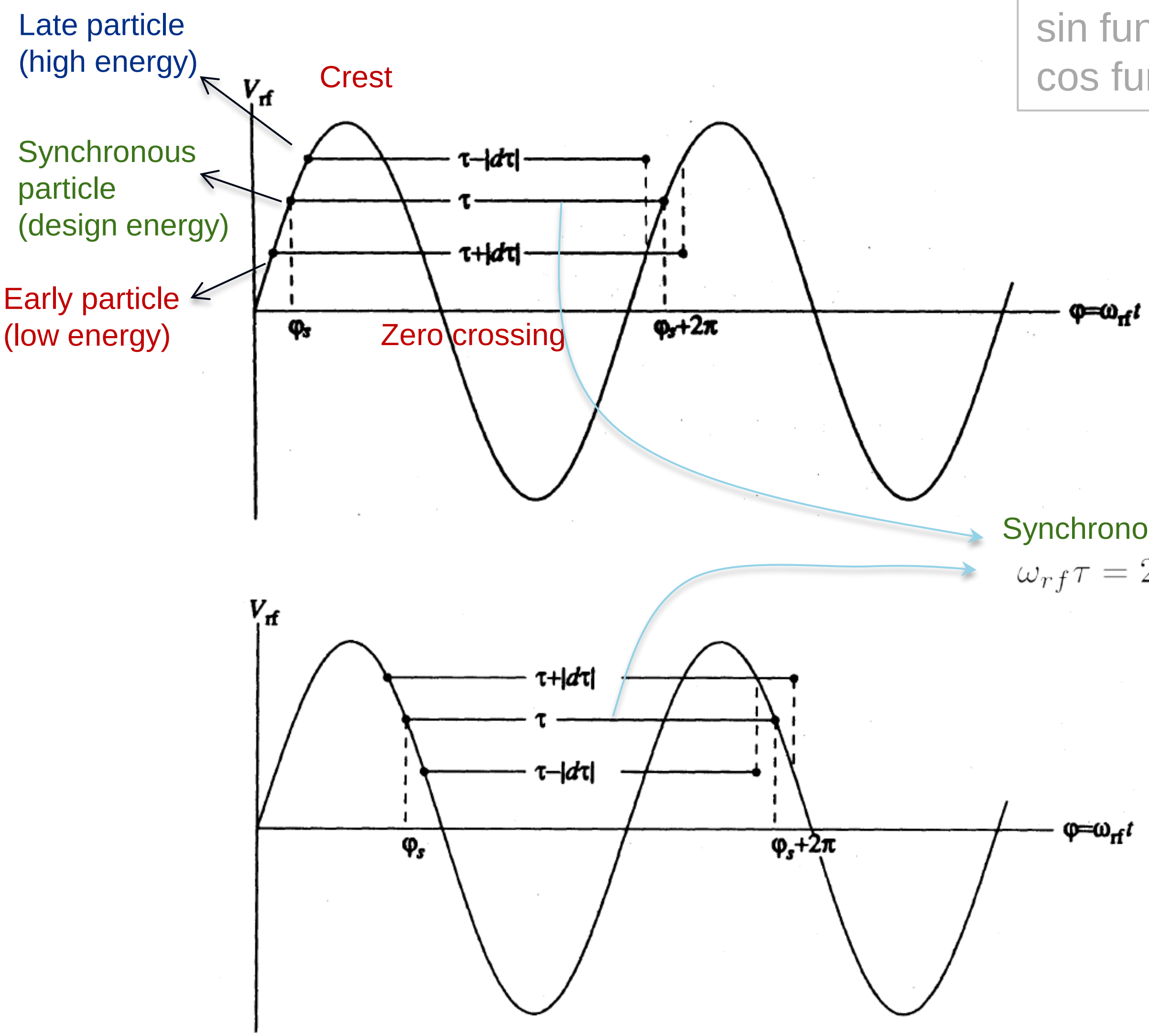
Note that there is a certain energy ($\gamma_0 = \gamma_{tr}$, called transition energy) at which the time dispersion vanishes, and all particle pass through the system in the same amount of time.

$$\eta_\tau = 0 = \alpha_c - \frac{1}{\gamma_0^2} = \alpha_c - \frac{1}{\gamma_{tr}^2}$$

- Below transition: $\boxed{\eta_\tau = \frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma_0^2} < 0, \quad \gamma_0 < \gamma_{tr}}$
  - Particles of higher momentum pass through the system more quickly, which is the natural state of affairs in linear systems.
- Above transition: $\boxed{\eta_\tau = \frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma_0^2} > 0, \quad \gamma_0 > \gamma_{tr}}$
  - Particles of higher momentum take more time to pass the system , since the added path length of a higher-momentum trajectory outweighs the added advantage in velocity, which becomes progressively smaller as particle becomes more relativistic.

[Example]

*Convention:
sin function for circular machines, and cos function for linear machines.



Late particle (high energy)

Crest

Synchronous particle (design energy)

Early particle (low energy)

Zero crossing

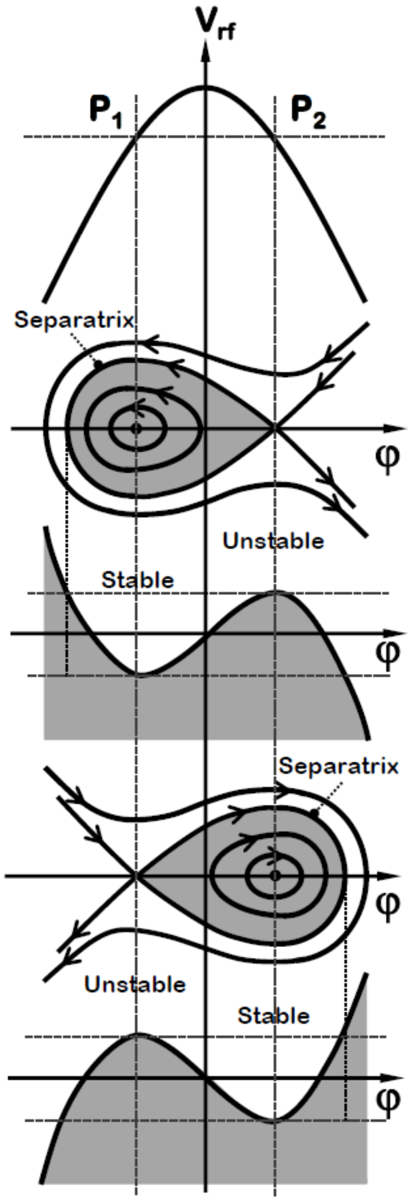
Below transition: stable oscillation for off crest with $\Delta\varphi < 0$

Synchronous particle arrives at the same voltage

$$\omega_{rf}T = 2\pi, \quad \text{or} \quad 2\pi h \quad (h = 1, 2, \cdots)$$

Above transition: stable oscillation for off crest with $\Delta\varphi > 0$

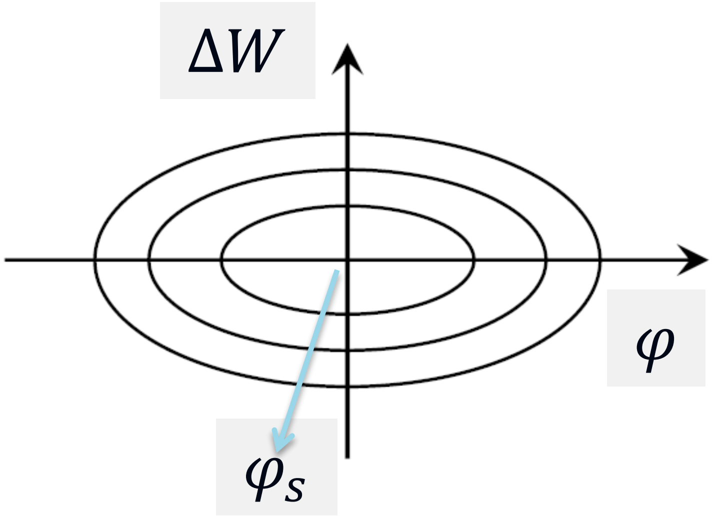
Synchrotron oscillation



Below transition

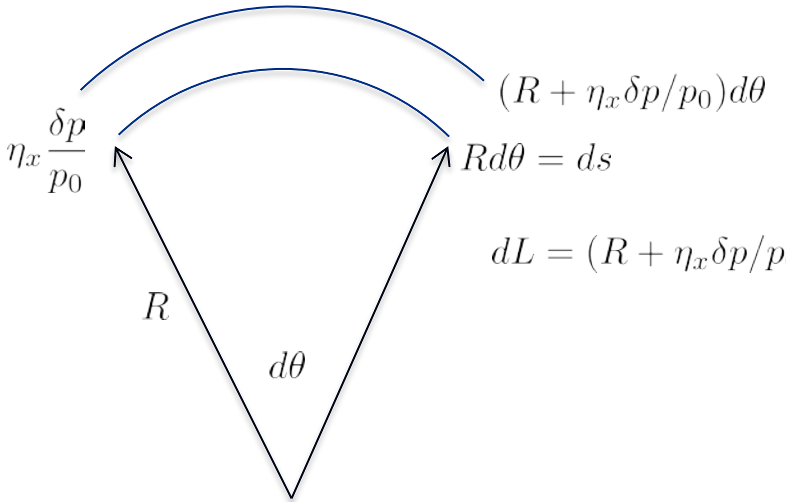
Above transition

SHO-like oscillation near the synchronous phase



Momentum compaction VS Dispersion

- Path length change around the circular path:



$$dL = (R + \eta_x \delta p/p_0) d\theta - R d\theta = \eta_x \delta p/p_0 d\theta$$

$$\delta L = \int dL = \frac{\delta p}{p_0} \int \eta_x d\theta = \frac{\delta p}{p_0} \int \frac{\eta_x}{R} ds$$

- For a single pass system:

$$\alpha_c = \frac{\delta L/L_0}{\delta p/p_0} = \frac{1}{s - s_0} \int_{s_0}^{s} \frac{\eta_x(\tilde{s})}{R(\tilde{s})} d\tilde{s}$$

- For a closed system:

$$\alpha_c = \frac{\delta L/L_0}{\delta p/p_0} = \frac{1}{C_0} \oint \frac{\eta_x(\tilde{s})}{R(\tilde{s})} d\tilde{s}$$

For a straight section,
$R \rightarrow \infty$, no contribution to the integral

In a storage ring, the momentum
compaction is usually positive
(but, in an anti-bend, can be negative)

Circumference of the design orbit

Chromaticity (or Chromatic aberration)

- Offsets of energy in the particles cause not only dispersion but also result in different focusing strengths of the magnetic elements:

$$\frac{qB'}{p} = \frac{qB'}{p_0(1+\delta_p)} = \frac{qB'}{p_0}(1-\delta_p) = k_1(1-\delta_p) = k_1 - k_1\delta_p$$

[Note] In fact, the weak focusing term from the dipole yields chromaticity as well. But usually, its contribution is “weaker” than from quadrupoles.
[Note] Chromatic aberration is a nonlinear effect ($\propto \delta_p x$).

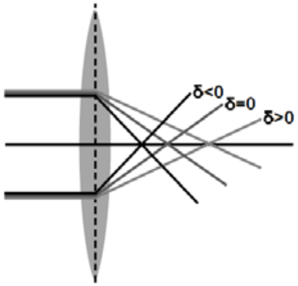


FIGURE 2.20
Chromaticity of a focusing quadrupole.

- The chromaticity is always negative: an increase in momentum always leads to a reduction in focusing strength.

$$\xi = \frac{1}{2\pi}\frac{d(\text{Phase advance})}{d\delta_p} = \frac{d(\text{Tune})}{d\delta_p} = \frac{d\nu}{d\delta_p}$$

Tune is proportional to the net focusing strength

[Note] In some literatures, the chromaticity is defined after normalization by the tune value.

- It is possible to reduce the chromaticity sufficiently using sextupoles.

Q: 포항가속기연구소의 PLS-II 저장링은 3 GeV로 운전이 됩니다. Momentum compaction factor 는 0.00138 이라고 가정하면, 이 장치는 below transition 인가요, 아니면 above transition 인가요?