Geometric Measure of Profile Similarity in High-Dimensional Phase Space

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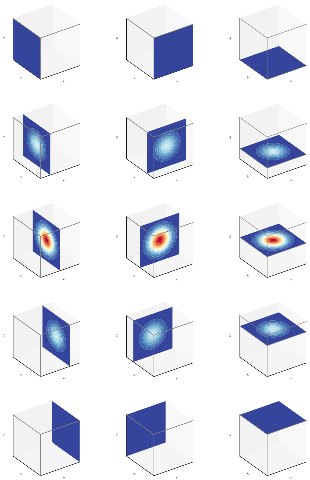
High-dimensional phase space tomography

It is easy to prove that there is sufficient low-dim information to reconstruct the high-dim distribution, but it is not clear how to choose a better measurement set.



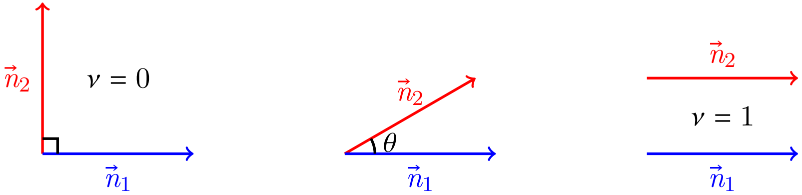
Evaluation of measurement quality:

- **Traditional method:** phase advance
  - Unclear effectiveness for coupling terms in high dimensions.
  - No guidance for measurements without transport.
- **New Perspective:** projection geometry
  - The geometric relationship among profiles determines information acquisition efficiency:
    - Closer profiles → higher redundancy
    - More orthogonal profiles → stronger complementarity
  - We can evaluate profile similarity by comparing their "angle".

Angle metric between 2D planes in 4D phase space

- We know that for two unit vectors:

$$(\vec{n}_1 \cdot \vec{n}_2)^2 = \cos^2\theta \overset{let}{=\!=} \nu \in [0,1]$$



- The value of ν reflects the magnitude of the angles between unit vectors.
- It can also reflect the angle between two 2D planes in 3D space, i.e., the squared dot product of their unit normal vectors.

Angle metric between 2D planes in 4D phase space

- For 2D planes in 4D space, we can adopt a similar approach, except that the normal vector is replaced by a 2D normal subspace.
- Search all unit vectors in both normal subspaces:

$$
\begin{aligned}
S &= \int_0^{2\pi}\int_0^{2\pi} \left[\vec{n}_1(\theta)\cdot\vec{n}_2(\phi)\right]^2 \,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_0^{2\pi}\int_0^{2\pi} \left[(\cos\theta\vec{e}_1+\sin\theta\vec{e}_2)\cdot(\cos\phi\vec{f}_1+\sin\phi\vec{f}_2)\right]^2 \mathrm{d}\theta\,\mathrm{d}\phi \\
&= \boxed{\pi^2\cdot\left[(\vec{e}_1\cdot\vec{f}_1)^2+(\vec{e}_1\cdot\vec{f}_2)^2+(\vec{e}_2\cdot\vec{f}_1)^2+(\vec{e}_2\cdot\vec{f}_2)^2\right]} \in [0, 2\pi^2]
\end{aligned}
$$

where

- $\vec{e}_1$ $\vec{e}_2$: an arbitrary orthonormal basis of the normal subspace of plane-1;
- $\vec{f}_1$ $\vec{f}_2$: an arbitrary orthonormal basis of the normal subspace of plane-2.

Angle metric between 2D planes in 4D phase space

- Thus, the total squared dot products of the basis vectors characterize the overall angle between the two normal subspaces (or the planes themselves).
- Define the normalized similarity metric for two 2D planes as:

$$\nu = \frac{S}{2\pi^2} = \boxed{\frac{1}{2} \cdot \left[(\vec{e}_1 \cdot \vec{f}_1)^2 + (\vec{e}_1 \cdot \vec{f}_2)^2 + (\vec{e}_2 \cdot \vec{f}_1)^2 + (\vec{e}_2 \cdot \vec{f}_2)^2\right]} \in [0, 1]$$

  - $\nu = 0$: the two planes are completely orthogonal;
  - $\nu = 1$: the two planes are completely coincident.

Generalization to N-D phase space | Formula Definition

- Projection subspace and normal subspace are complementary:

$$N = n_p + n_n \text{ (projection + normal)}.$$

- Two definitions for the distance between two profiles:
  - Based on an orthonormal basis of the normal subspace of the plane:

$$\boxed{\nu_n = \frac{1}{n_n} \sum_{i=1}^{n_n} \sum_{j=1}^{n_n} (\vec{e}_i \cdot \vec{f}_j)^2}$$

  - Based on an orthonormal basis of the plane itself:

$$\boxed{\nu_p = \frac{1}{n_p} \sum_{i=n_{n+1}}^{N} \sum_{j=n_{n+1}}^{N} (\vec{e}_i \cdot \vec{f}_j)^2}$$

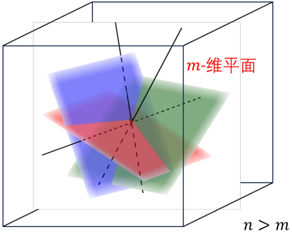
Generalization to N-D phase space | Formula Selection

- ν_n **and** ν_p **are complementary**, and their relation is as follows:

$$\boxed{\nu_n = 1 - \frac{n_p}{n_n}(1 - \nu_p)}$$

or equivalently,

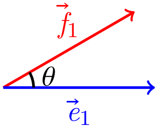
$$\boxed{\nu_p = 1 - \frac{n_n}{n_p}(1 - \nu_n)}$$



- **When** $n_n \neq n_p$, ν_n **and** ν_p **cannot both be zero** (complete orthogonality):
  - If $N < 2n_n$: normal subspaces cannot be orthogonal → $\nu_n > 0$ always;
  - If $N < 2n_p$: planes themselves cannot be orthogonal → $\nu_p > 0$ always.
- **Metric selection depends on dimensionality:** Choose the metric with $N > 2n_n$ or $N > 2n_p$ to allow zero and clearly distinguish orthogonal vs. non-orthogonal.

Similarity metric for 3D profiles in 4D phase space

1. **Choosing the formula:** $N = 4, n_p = 3, n_n = 1 \rightarrow N > 2n_n \xrightarrow{use} \nu_n = (\vec{e}_1 \cdot \vec{f}_1)^2$



   Problem reduces to squared dot product of two unit vectors.

2. **Find $\vec{e}_1$ and $\vec{f}_1$:**
   - Note: $\vec{e}_1$ and $\vec{f}_1$ should vary with the beam matrix.
   - Compute the 4×4 covariance matrix Σ of the 3D profile in 4D phase space, with $rank(\Sigma) = 3$.
   - Solve for the eigenvalues and eigenvectors of Σ. The eigenvector corresponding to the zero eigenvalue is the unit normal vector of the 3D profile.

3. **Pairwise compare to determine the best combination.**

Thanks!