Quantifying Differences between High-Dimensional Beam Phase Space Distributions with f-Divergences

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Contents

Quantification of Beam Distribution Differences

- **Common Method:** Extract parameters from distributions for comparison.

Mismatch Factor
RMS Emittance
99 % Emittance

} —Shortcomings→

- Identical parameters do not guarantee identical distributions.
- As the dimensionality increases, parameter comparison becomes more complex and less accurate.

- **Improved Method:** Directly compare the total distribution, We adopt

f-divergences.

- Provide reasonable computational complexity.
- Can obtain comprehensive difference information.

f-Divergences

- The f-divergences are a common class of methods used to measure the difference between two distributions, defined as follows:

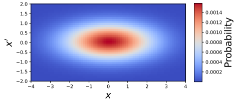
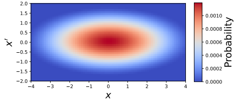
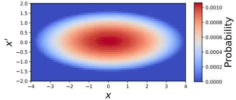
$$D_f[\rho_1(\mathbf{x}) \parallel \rho_2(\mathbf{x})] = \int \rho_2(\mathbf{x})\, f\left[\frac{\rho_1(\mathbf{x})}{\rho_2(\mathbf{x})}\right] \mathrm{d}\mathbf{x},$$

- $f(\cdot)$ is a convex function and satisfies $f(1) = 0$;
- Different $f(\cdot)$ correspond to different statistical divergences:

Name	$f(t)$	h	**Range**
Kullback-Leibler	$t \ln t$	$\rho_1(\mathbf{x}) \ln \left[\frac{\rho_1(\mathbf{x})}{\rho_2(\mathbf{x})}\right]$	$[0, \infty)$
Jensen-Shannon	$\frac{1}{2}\left[\ln\left(\frac{2}{t+1}\right)^{t+1} + t \ln t\right]$	$\frac{1}{2}\left\{\rho_2(\mathbf{x}) \ln\left[\frac{2\rho_2(\mathbf{x})}{\rho_1(\mathbf{x})+\rho_2(\mathbf{x})}\right] + \rho_1(\mathbf{x}) \ln\left[\frac{2\rho_1(\mathbf{x})}{\rho_1(\mathbf{x})+\rho_2(\mathbf{x})}\right]\right\}$	$[0, \ln 2]$
Total Variation	$\frac{1}{2}\lvert t-1 \rvert$	$\frac{1}{2}\lvert \rho_1(\mathbf{x}) - \rho_2(\mathbf{x}) \rvert$	$[0, 1]$
Squared Hellinger	$\frac{1}{2}(\sqrt{t}-1)^2$	$\frac{1}{2}\left[\sqrt{\rho_1(\mathbf{x})} - \sqrt{\rho_2(\mathbf{x})}\right]^2$	$[0, 1]$

Four forms of f-divergences; $t = \rho_1(\mathbf{x})/\rho_2(\mathbf{x})$; $h = \rho_2(\mathbf{x}) \cdot f[\rho_1(\mathbf{x})/\rho_2(\mathbf{x})]$; $\mathbf{x} \in \mathbb{R}^{2n}$

Several Common 4D Beam Distributions

Distribution with Elliptical Symmetry	**Definition** $\rho(x, x', y, y') = \rho(I)$ $I = \mathbf{x}^T \Sigma^{-1} \mathbf{x}$	**Schematic Diagram of 2D Projection** (x, x')
Gaussian	$\frac{1}{(\sqrt{2\pi})^4 \lvert\Sigma\rvert^{\frac{1}{2}}} \cdot e^{-\frac{1}{2} I}$	
Parabolic	$\frac{6}{(\sqrt{8\pi})^4 \lvert\Sigma\rvert^{\frac{1}{2}}} \cdot \left(1 - \frac{I}{8}\right), \quad I < 8$	
Waterbag	$\frac{2}{(\sqrt{6\pi})^4 \lvert\Sigma\rvert^{\frac{1}{2}}}, \quad I < 6$	

Quadratic form expression of beam distributions with elliptical symmetry; $\mathbf{x} = (x, x', y, y')^T$*;*
Σ *is a covariance matrix composed of 10 independent second-order moments.*

The D_f Between Different Types of Distributions with Identical Σ

Theorem

In the 2n-D phase space $(x_1, x'_1, \cdots, x_n, x'_n)$, for any two elliptically symmetric beam distributions of different types sharing identical covariance matrix $\mathbf{\Sigma}$, their f-divergences is independent of the specific values of the second-order moments in $\mathbf{\Sigma}$.

- The f-divergence values between different 4D distributions sharing identical $\mathbf{\Sigma}$:

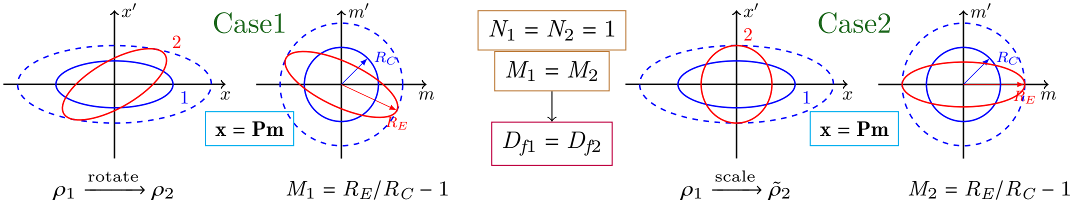
$\rho_1(\mathbf{x}) - \rho_2(\mathbf{x})$	D_{KL}	D_{JS}	D_{TV}	D_{Hel}
Parabolic – Gaussian	0.185837	0.054823	0.226909	0.262902
Waterbag – Gaussian	0.495922	0.134071	0.391299	0.407679
Waterbag – Parabolic	0.231856	0.071380	0.223872	0.309006

- According to this theorem, as long as the Σ of the two distributions is the same, the above results are fixed.

f-Divergences in Relation to Mismatch Factors

Theorem

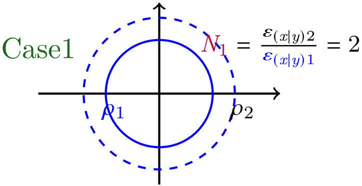
In the $(x_1, x_1', x_2, x_2', \cdots, x_i, x_i', \cdots, x_n, x_n')$ *phase space (2n-D), given two elliptically symmetric and uncoupled beam distributions in the* $\{x_i \,|\, i = 1, 2, \cdots, n\}$ *motion directions, if their RMS emittances* $\{\varepsilon_i \,|\, i = 1, 2, \cdots, n\}$ *in the 2D subspaces* $\{(x_i, x_i') \,|\, i = 1, 2, \cdots, n\}$ *are identical, then the f-divergence between them depends solely on the mismatch factors* $\{M_i \,|\, i = 1, 2, \cdots, n\}$ *in these 2D subspaces.*

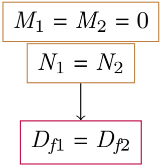


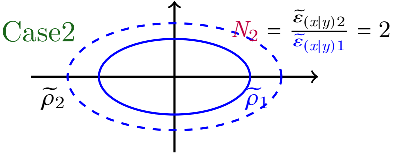
f-Divergences in Relation to RMS Emittance Scaling Factors

Theorem

In the $(x_1, x_1', x_2, x_2', \cdots, x_i, x_i', \cdots, x_n, x_n')$ phase space (2n-D), given two elliptically symmetric and uncoupled beam distributions in the $\{x_i | i = 1, 2, \cdots, n\}$ motion directions, if the mismatch factors $\{M_i | i = 1, 2, \cdots, n\}$ in the 2D subspaces represented by $\{(x_i, x_i') | i = 1, 2, \cdots, n\}$ are all zero, then the f-divergence between them depends solely on the RMS emittance scaling factors $\{N_i | i = 1, 2, \cdots, n\}$ in these 2D subspaces.







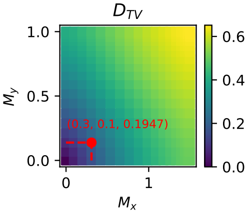
Assessment Standards for f-Divergences

- We used the previous correspondences and three ideal distributions to establish assessment standards for the f-divergence.
- This can roughly explain the source of the differences described by f-divergence values.

For example: Beam 4D transverse phase space tomography result analysis

ρ_1 closest to ↓ Gaussian | Real distribution ⟷ $D_{TV} = 0.1947$ ⟷ Reconstructed distribution | ρ_2 ↓ closest to Gaussian

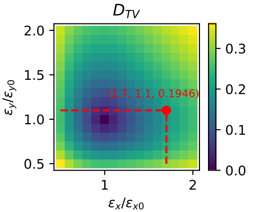
Case 1 (Gaussian-Gaussian):



$$\frac{\varepsilon_x}{\varepsilon_{x0}} = \frac{\varepsilon_y}{\varepsilon_{y0}} = 1$$

$$M_x = 0.3(or\ 0.1)$$
$$M_y = 0.1(or\ 0.3)$$

Case 2 (Gaussian-Gaussian):

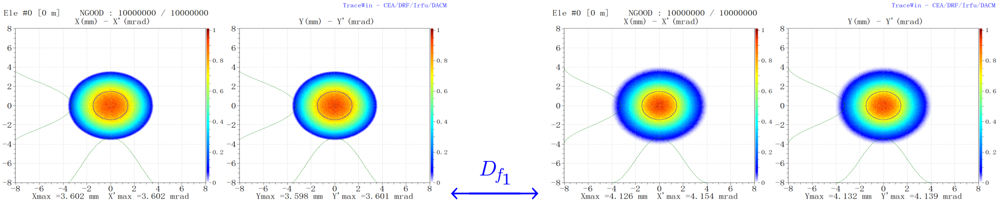


$$M_x = M_y = 0$$

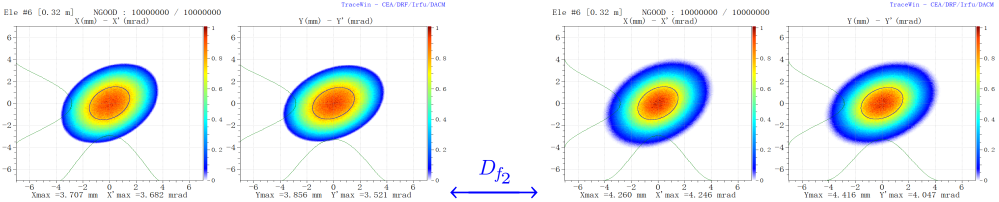
$$\frac{\varepsilon_x}{\varepsilon_{x0}} = 1.7(or\ \frac{1}{1.7})$$
$$\frac{\varepsilon_y}{\varepsilon_{y0}} = 1.1(or\ \frac{1}{1.1})$$

Dynamic Transport Evolution of D_f | Linear Transport

f-Divergence remains invariant during linear transport $(D_{f_1} = D_{f_2})$



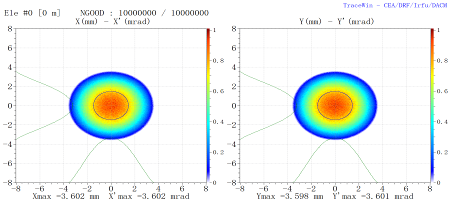
(a) Initial waterbag distribution (b) Initial parabolic distribution



(c) Quadrupole transport (waterbag) (d) Quadrupole transport (parabolic)

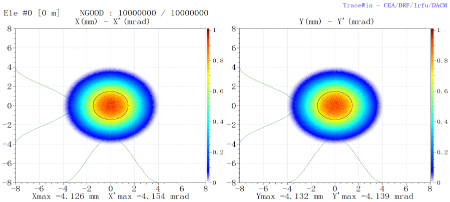
Dynamic Transport Evolution of D_f | Nonlinear Transport

f-Divergence also remains invariant during nonlinear transport $(D_{f_1} = D_{f_3})$

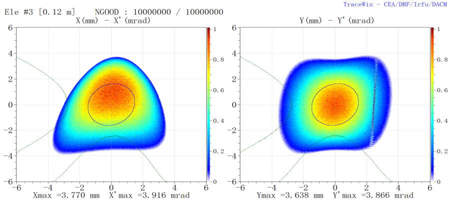


(e) Initial waterbag distribution

D_{f_1} ⟷

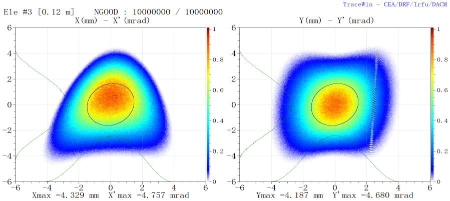


(f) Initial parabolic distribution



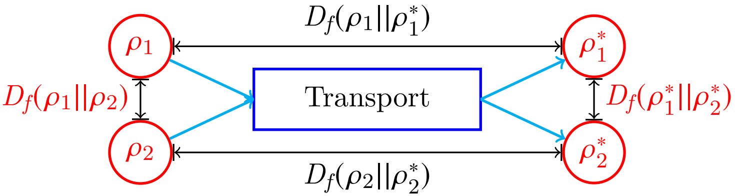
(g) Sextupole transport (waterbag)

D_{f_3} ⟷



(h) Sextupole transport (parabolic)

Dynamic Transport Evolution of D_f | Space Charge Forces



- Under linear and nonlinear transport: $D_f(\rho_1||\rho_2) = D_f(\rho_1^*||\rho_2^*)$.
- Space charge forces can break this conservation.

There is still much to explore about these properties.

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Thanks!