SOFT: Single-Optics Four-dimensional phase space Tomography via double-rotated 2D measurements

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Background

Detailed knowledge of high-dimensional phase space distributions is crucial for predicting and controlling the evolution of intense beams.

Possible Beam Coupling

$$\sigma = \begin{pmatrix} \langle xx \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle & \langle yx' \rangle & \langle x'y' \rangle \\ \langle xy \rangle & \langle yx' \rangle & \langle yy \rangle & \langle yy' \rangle \\ \langle xy' \rangle & \langle x'y' \rangle & \langle yy' \rangle & \langle y'y' \rangle \end{pmatrix}$$

4D Transverse Phase-Space σ Matrix

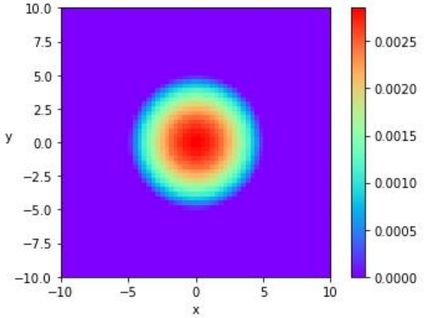
$$\begin{pmatrix} \langle xy \rangle & \langle xy' \rangle \\ \langle yx' \rangle & \langle x'y' \rangle \end{pmatrix}$$

- if all elements are zero: uncoupled beam
- if any elements is nonzero: uncoupled beam

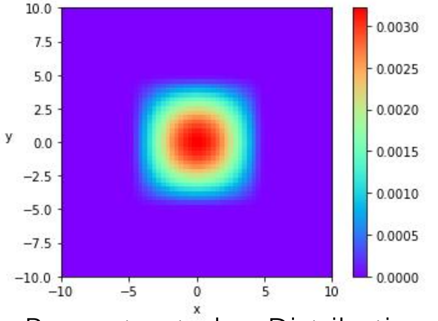
Possible causes of beam coupling:

1) ECR ion source
2) Asymmetric beam passing through a solenoid
3) ……

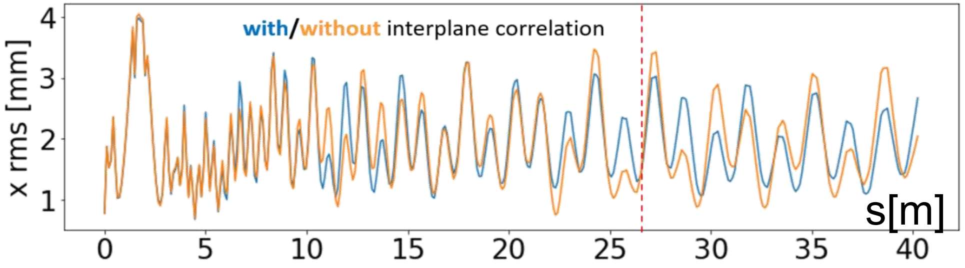
No Coupling ≠ Directional Independence



true xy distribution



Reconstructed xy Distribution Assuming x–y Independence

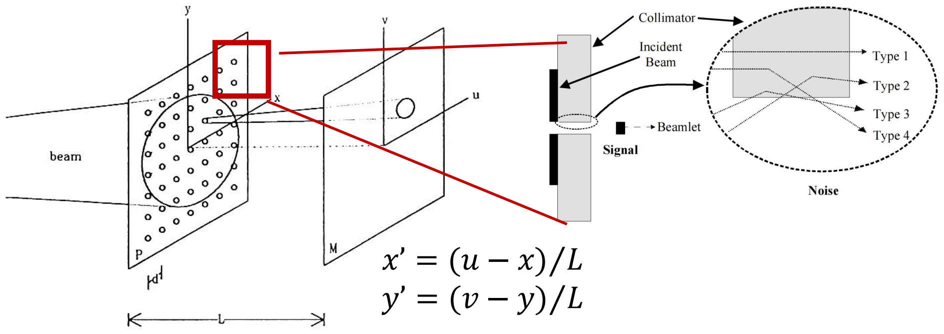


RMS Variation Along Longitudinal Direction: x–y Independent vs. True xy Distribution

$\rho_{4D} \neq \rho_{2D}(x, x') \cdot \rho_{2D}(y, y')$, assuming equality causes a significant loss of information.

Single-optics 4D measurement

■ Pepper-pot



$$x' = (u - x)/L$$
$$y' = (v - y)/L$$

Concept of the “pepper-pot” method to measure transverse emittance.

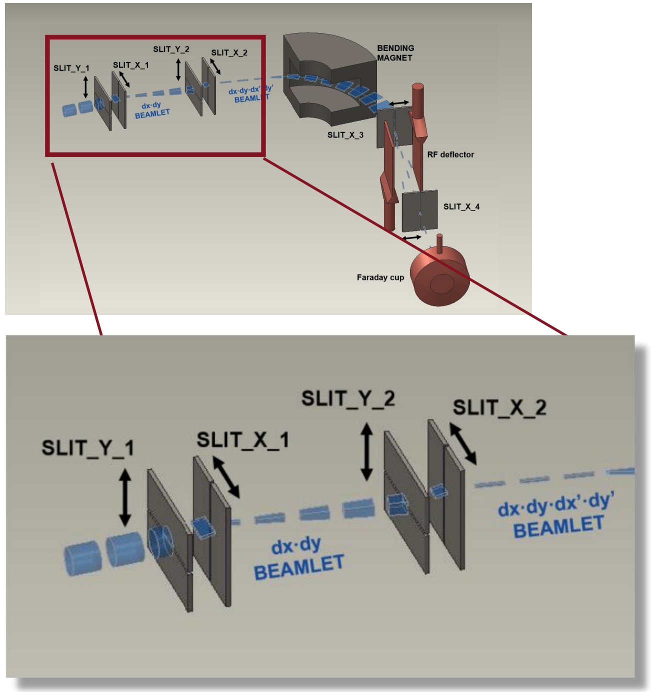
Advantages：

- Low cost: simple measurement principle
- Time-efficient: single-optics measurement

Limitations：

- Resolution limitation
- **Discontinuous sampling of phase space, Incomplete beam distribution**
- Particle scattering while passing through the mask

■ Direct measurement



SNS 6D measurement arrangement

Estimated Scanning Time

Number of steps per degree of freedom, n is taken as 10–20 typically

$$N_{bins} = n^4$$

Total number of bins

$$T_{total} = 20^4 sec \approx 1.85 day$$

Compared with slit measurements, performing eight 2D scans:

$$T_{total} = 8 \times 20^2 sec = 0.89h$$

The scanning time differs by a factor of 50.

- *Complete beam information can be obtained*
- ***Excessive time consumption*** *limits its use to research rather than practical operation*

4D Phase Space Reconstruction:
From Low-Dimensional Projections to High-Dimensional Distributions

Measurement Method: Efficient Information Acquisition in 2D

Analogy: CT Scanning

Reconstruction Algorithm: Reconstructing the Beam from Scan Results

Multi-Optics 4D Measurement

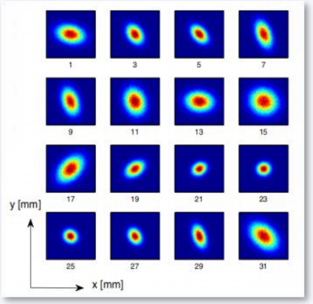
Challenge:
cross plane information

$$\sigma = \begin{pmatrix} \langle xx\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle xx'\rangle & \langle x'x'\rangle & \langle yx'\rangle & \langle x'y'\rangle \\ \langle xy\rangle & \langle yx'\rangle & \langle yy\rangle & \langle yy'\rangle \\ \langle xy'\rangle & \langle x'y'\rangle & \langle yy'\rangle & \langle y'y'\rangle \end{pmatrix}$$

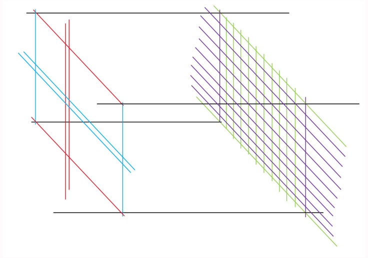
4D Transverse Phase-Space σ Matrix

■ *Quad Scan & Fluorescent Screen Measurement*

Obtaining <xy> Using Fluorescent Screen Measurement

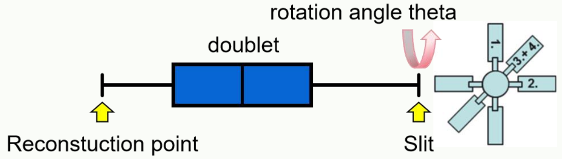


■ *Perpendicular scans*



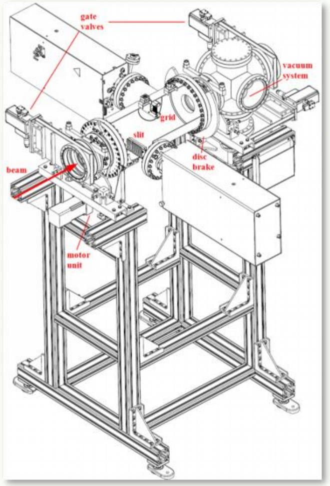
Obtaining up to Two Coupling Terms

■ *ROSE: Rotating system for 4D transverse measurements*



Parallel Rotation of Slit and Grid

- ✓ $\langle xy\rangle_f^{a,b}, \langle x'y'\rangle_f^{a,b}$ **obtainable**
- × $\langle xy'\rangle_f^{a,b}, \langle x'y\rangle_f^{a,b}$ **unobtainable**
- × **Still Requires Multi-optics**



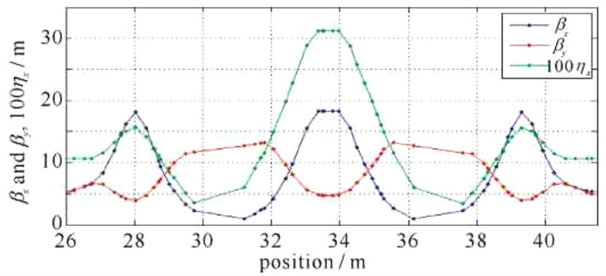
Summary of Multi-Optics Measurement

- Development Trend: Increasingly More Information Accessible, but Complete Information Still Unobtainable in a Single Optics Measurement
- Reason for Multi-Optics: **Insufficient Coupling Information under a Single Optics Condition**

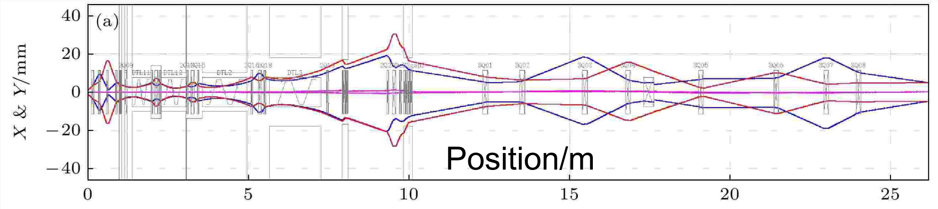
Multi-Optics vs Single-Optics Measurement

Limitations of Multi-Optics Measurement

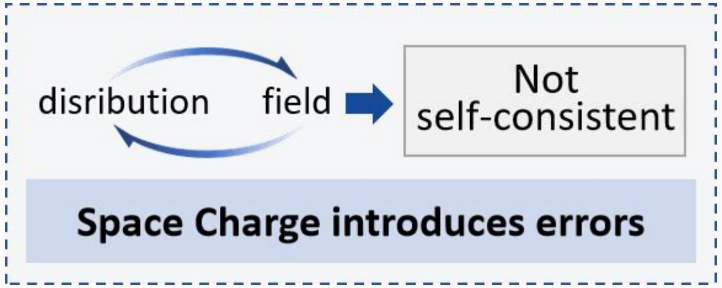
Nonlinearity and Uncertainty



Beam Loss



Space Charge



Multi-Optics Measurement:

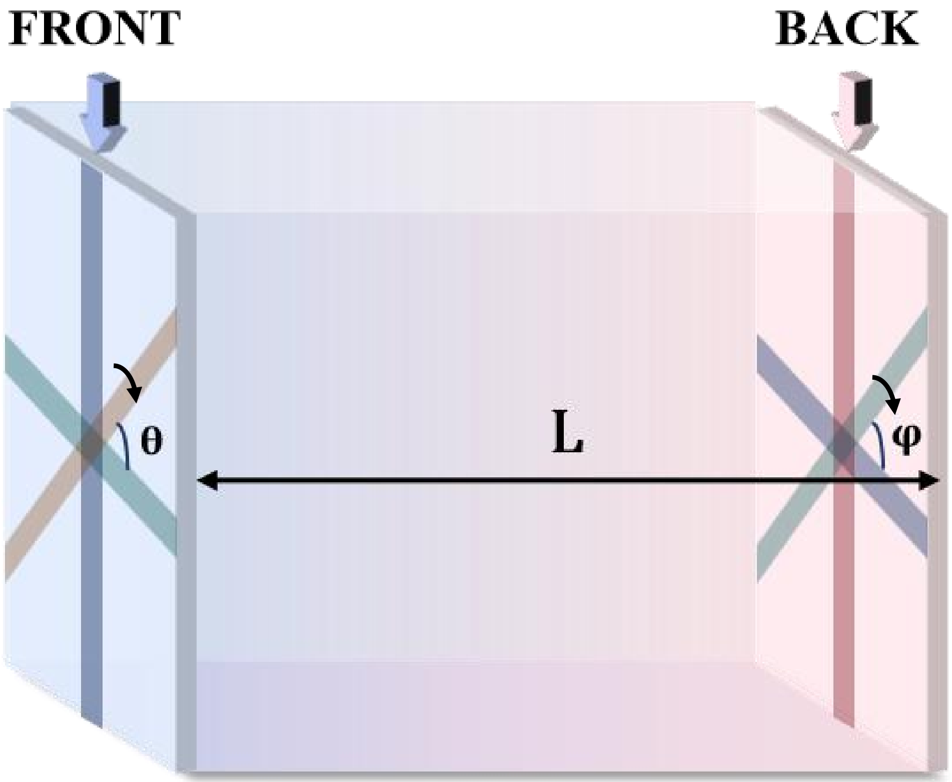
Single-Optics Measurement:

Reconstruction–Measurement Point Coincide

Advantages of Single-Optics Measurement

- **More effective information obtainable**
- Avoids most transport-related issues
- Potentially **shorter measurement time**

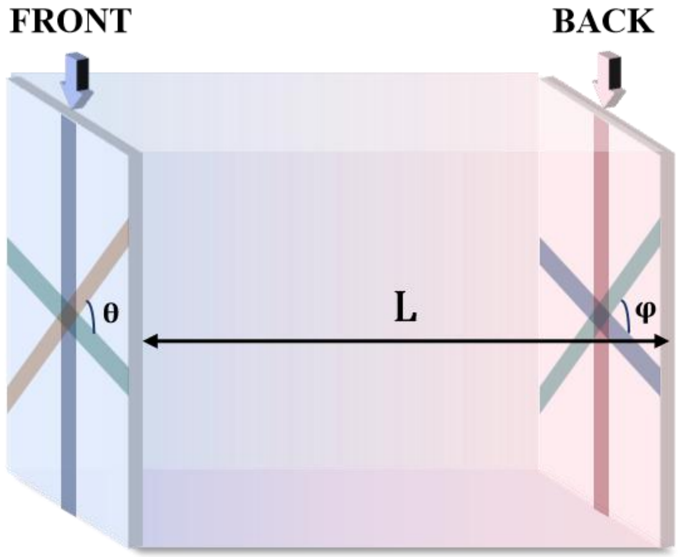
Measurement Principle of SOFT



Schematic of SOFT

SOFT conducts 2D phase space scans with two independently rotatable slits to extract otherwise inaccessible cross-plane information.

Measurement Principle of SOFT



Schematic of SOFT

$$\sigma = \begin{pmatrix} \langle xx\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle xx'\rangle & \langle x'x'\rangle & \langle yx'\rangle & \langle x'y'\rangle \\ \langle xy\rangle & \langle yx'\rangle & \langle yy\rangle & \langle yy'\rangle \\ \langle xy'\rangle & \langle x'y'\rangle & \langle yy'\rangle & \langle y'y'\rangle \end{pmatrix}$$

4D Transverse Phase-Space σ Matrix

Prove using linear algebra

Transfer Matrix within the Device

$$\begin{bmatrix} x_{front} \\ x_{back} \\ y_{front} \\ y_{back} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & L & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & L \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

Rotating Matrix

$$\begin{bmatrix} u_{front} \\ u_{back} \\ v_{front} \\ v_{back} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\varphi & 0 & -\sin\varphi \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} x_{front} \\ x_{back} \\ y_{front} \\ y_{back} \end{bmatrix}$$

$$\begin{bmatrix} u_{front} \\ u_{back} \\ v_{front} \\ v_{back} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ \cos\varphi & L\cdot\cos\varphi & -\sin\varphi & -L\cdot\sin\varphi \\ \sin\theta & 0 & \cos\theta & 0 \\ \sin\varphi & L\cdot\sin\varphi & \cos\varphi & L\cdot\cos\varphi \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

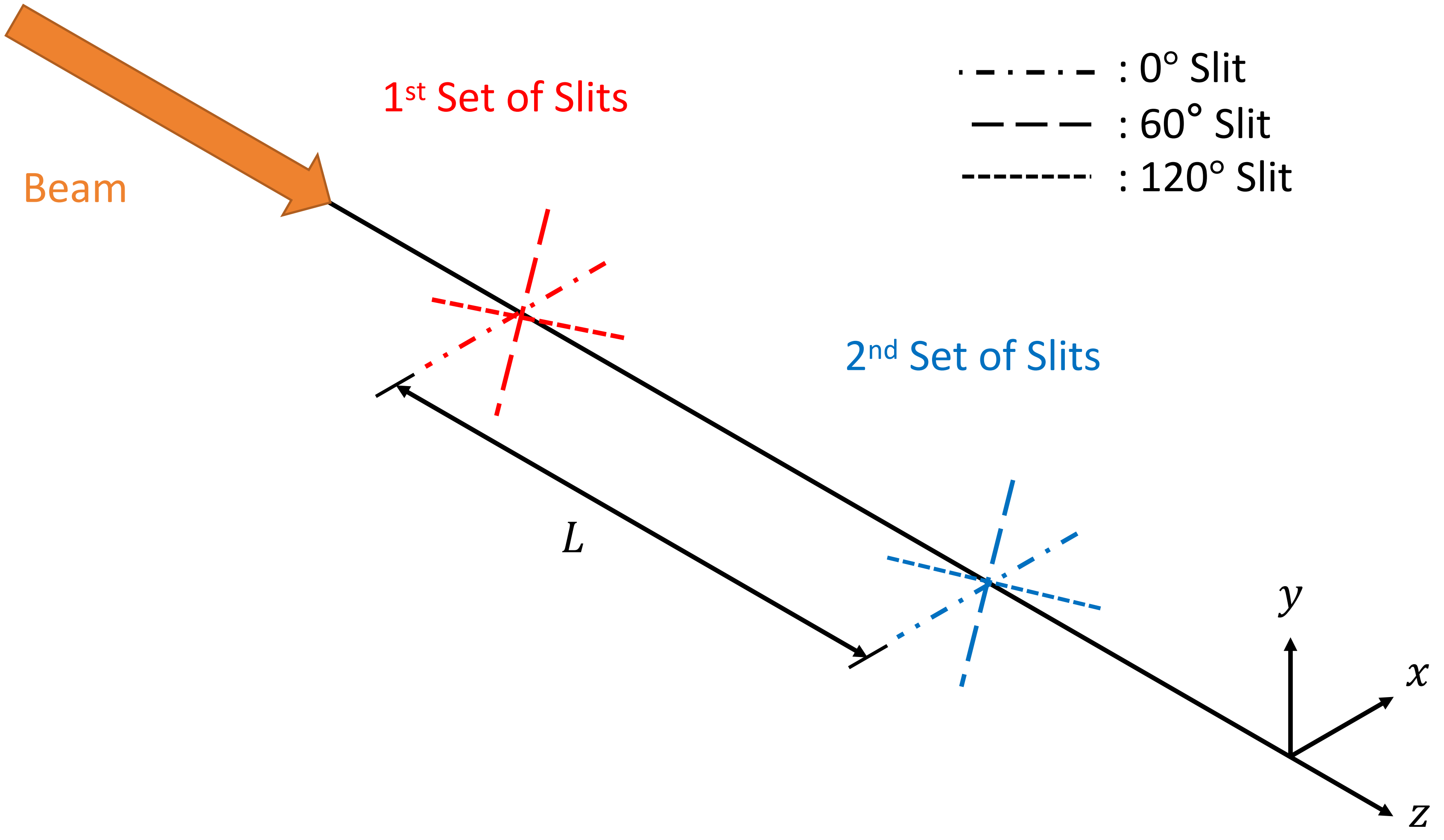
$$\langle u_f u_f\rangle = \cos^2\theta\langle xx\rangle - \sin 2\theta\,\langle xy\rangle + \sin^2\theta\langle yy\rangle$$

$$\langle u_f u_b\rangle = \cos\theta\cos\varphi\,(\langle xx\rangle + L\langle xx'\rangle) - \sin\theta\cos\varphi\,(\langle xy\rangle + L\langle x'y\rangle) - \cos\theta\sin\varphi\,(\langle xy\rangle + L\langle xy'\rangle) + \sin\theta\sin\varphi\,(\langle yy\rangle + L\langle yy'\rangle)$$

$$\langle u_b u_b\rangle = \cos^2\varphi(\langle xx\rangle + 2L\langle xx'\rangle + L^2\langle x'x'\rangle) - \sin 2\varphi\,[\langle xy\rangle + L^2\langle x'y'\rangle] - \sin 2\varphi\,[L(\langle xy'\rangle + \langle x'y\rangle)] + \sin^2\varphi(\langle yy\rangle + 2L\langle yy'\rangle + L^2\langle y'y'\rangle)$$

Reconstruction Conditions: ≥3 θ Values and ≥3 φ Values

Simple SOFT: Add Slanted Slits

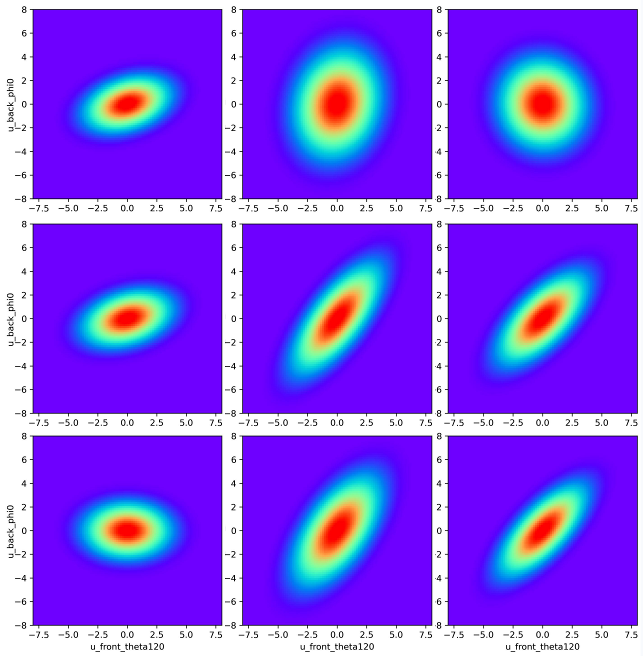


Verification of reconstruction feasibility

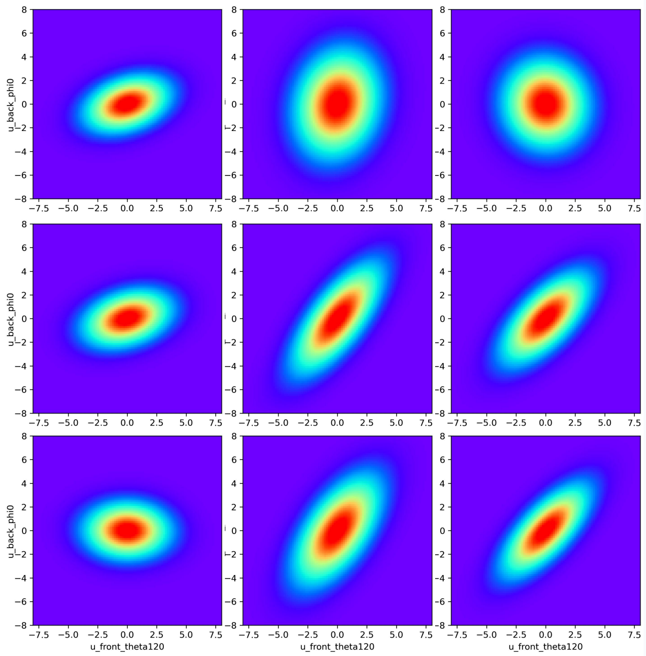
Reconstruction and reprojection using 9 measured projections based on Liwen Liu's maximum entropy algorithm*.

Comparison between Measured and Reprojected Data at the Same Angles

9 Measured Projections



Reprojected Results
at the Same Angles after Reconstruction



The reconstruction results also verify the feasibility of the SOFT principle.

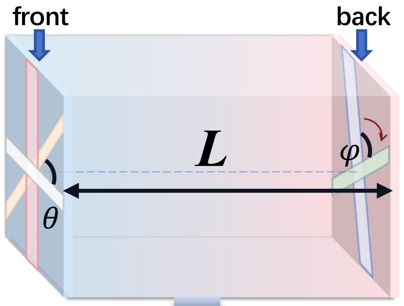
****TALK,*** *Liwen Liu et al.*

We proved SOFT works, how to make it more accurate?

1) Design Device
2) Choice of measurement parameters

Distance L Between Slits Determines the Error Sensitivity

Previous emittance scans: distance L only affects the angular resolution



$$Ax = b$$

Using the condition number to quantify the sensitivity of errors

↓

How L influence the cond(A)?

↓

$$cond(A) = \sqrt{\frac{\lambda_{max}(A^T A)}{\lambda_{min}(A^T A)}}$$

↓

a) Calculate elements of $A^T A$
b) How eigenvalue of $A^T A$ vary with m,n, L

$$A^TA = \begin{bmatrix} mn & 0 & \frac{1}{4}mn & Lmn & 0 & 0 & \frac{1}{4}Lmn & \frac{3}{8}L^2mn & 0 & \frac{1}{8}L^2mn \\ 0 & \frac{3}{2}mn & 0 & 0 & \frac{3}{4}Lmn & \frac{3}{4}Lmn & 0 & 0 & \frac{1}{2}L^2mn & 0 \\ \frac{1}{4}mn & 0 & mn & \frac{1}{4}Lmn & 0 & 0 & Lmn & \frac{1}{8}L^2mn & 0 & \frac{3}{8}L^2mn \\ Lmn & 0 & \frac{1}{4}Lmn & \frac{7}{4}L^2mn & 0 & 0 & \frac{1}{2}L^2mn & \frac{3}{4}L^3mn & 0 & \frac{1}{4}L^3mn \\ 0 & \frac{3}{4}Lmn & 0 & 0 & \frac{3}{4}L^2mn & \frac{1}{2}L^2mn & 0 & 0 & \frac{1}{2}L^3mn & 0 \\ 0 & \frac{3}{4}Lmn & 0 & 0 & \frac{1}{2}L^2mn & \frac{3}{4}L^2mn & 0 & 0 & \frac{1}{2}L^3mn & 0 \\ \frac{1}{4}Lmn & 0 & Lmn & \frac{1}{2}L^2mn & 0 & 0 & \frac{7}{4}L^2mn & \frac{1}{4}L^3mn & 0 & \frac{3}{4}L^3mn \\ \frac{3}{8}L^2mn & 0 & \frac{1}{8}L^2mn & \frac{3}{4}L^3mn & 0 & 0 & \frac{1}{4}L^3mn & \frac{3}{8}L^4mn & 0 & \frac{1}{8}L^4mn \\ 0 & \frac{1}{2}L^2mn & 0 & 0 & \frac{1}{2}L^3mn & \frac{1}{2}L^3mn & 0 & 0 & \frac{1}{2}L^4mn & 0 \\ \frac{1}{8}L^2mn & 0 & \frac{3}{8}L^2mn & \frac{1}{4}L^3mn & 0 & 0 & \frac{3}{4}L^3mn & \frac{1}{8}L^4mn & 0 & \frac{3}{8}L^4mn \end{bmatrix}$$

Matrix $A^T A$

m : the number of front slits
n : the number of back slits

Eigenvalues of the Matrix $A^T A$

$$\lambda_1 = \frac{1}{4}Lmn$$

$$\lambda_2, \lambda_3, \lambda_4 : \frac{1}{8}mn(Poly1, Root1), \frac{1}{8}mn(Poly1, Root2), \frac{1}{8}mn(Poly1, Root3)$$

$$\lambda_5, \lambda_6, \lambda_7 : \frac{1}{8}mn(Poly2, Root1), \frac{1}{8}mn(Poly2, Root2), \frac{1}{8}mn(Poly2, Root3)$$

$$\lambda_8, \lambda_9, \lambda_{10} : \frac{1}{8}mn(Poly3, Root1), \frac{1}{8}mn(Poly3, Root2), \frac{1}{8}mn(Poly3, Root3)$$

(Standard form: $a\lambda^3 + b\lambda^2 + c\lambda + d = 0$) :

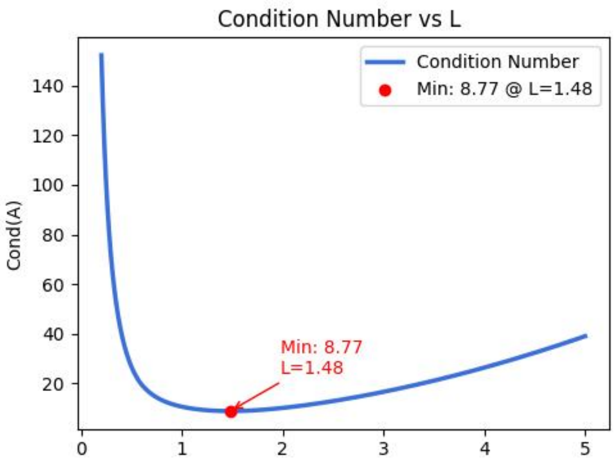
Equ1: $\lambda^3 - (4L^4 + 18L^2 + 10)\lambda^2 + (8L^6 + 24L^4 + 80L^2)\lambda - 32L^6 = 0$

Equ2: $\lambda^3 - (4L^4 + 10L^2 + 12)\lambda^2 + (8L^6 + 32L^4 + 48L^2)\lambda - 32L^6 = 0$

Equ3: $\lambda^3 - (2L^4 + 10L^2 + 6)\lambda^2 + (4L^6 + 8L^4 + 24L^2)\lambda - 8L^6 = 0$

$cond(A)$ is only related to L

For the idealized spherical beam, $cond(A)$ reaches its minimum when the distance at $L\frac{\sqrt{\langle x'^2\rangle}}{\sqrt{\langle x^2\rangle}} = 1.48$.



Given an arbitrary 4D beam distribution, which set of measurement angles minimzie the error?

Exhaustive search

Objective: minimize the condition number

$1°$ *step and 9 sets of angles*

over $C_{180^2}^{9} = 10^{35}$ combinations

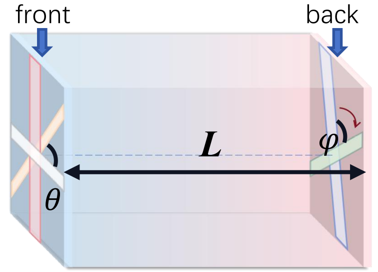
Computationally Impractical

Selection of Measurement Angles

If beam is a 4D sphere

By symmetry: Uniformly Distributed Projection Angles

Best Angles List
- theta_list: [0°, 60°, 120°]
- phi_list: [0°, 60°, 120°]



If beam is NOT a 4D sphere: try to mimic the symmetric case

Find the Optimal Measurement Angles: Closely Approximate those under Ideal Conditions

Transform RMS ellipsoid into sphere in transformed space

Coordinate Transformation

$$\boldsymbol{X}^T A \boldsymbol{X} = 1 \xrightarrow{\boldsymbol{X} = P\boldsymbol{Y}} \boldsymbol{Y}^T \boldsymbol{Y} = 1$$

$$\boldsymbol{X} = P\boldsymbol{Y} = Q\Lambda^{-1/2}\boldsymbol{Y}$$

inverse transformation:

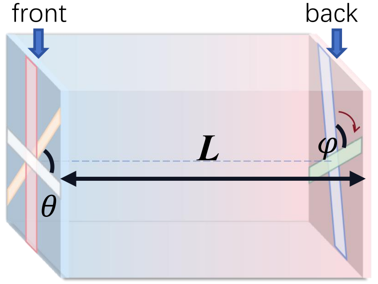
$$\boldsymbol{Y} = P^{-1}\boldsymbol{X} = \Lambda^{1/2}Q^T\boldsymbol{Y}$$

- Matrix A can be decomposed as $A = Q\Lambda Q^T$
- Q satisfying $\boldsymbol{Q}^T\boldsymbol{Q} = I$
- $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

Align Data into a Common Reference Frame for Analysis

New Metric: 4D Geometric Tool

- What $\theta - \varphi$ in lab space corresponds to uniform projection angles in transformed space?
- Each measurement corresponds to a 2D projection
- ***Question:*** how to quantify the parallelness between two 2D projections of a 4D distribution



Spatial "Angles": 4D Geometric Tool

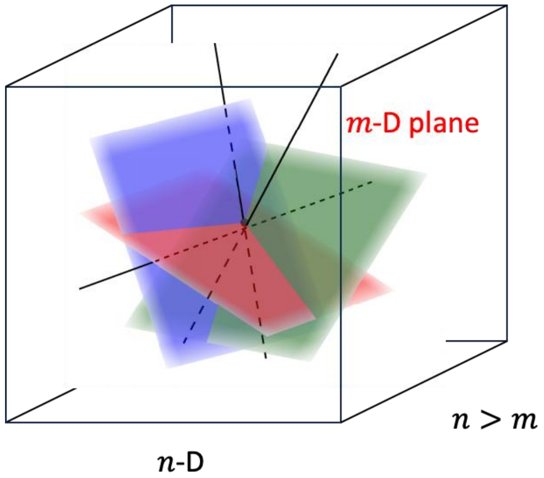
For two 2D planes in 4D space, the metric is defined as:

$$v = \frac{1}{2}[(\boldsymbol{e_1} \cdot \boldsymbol{f_1})^2 + (\boldsymbol{e_1} \cdot \boldsymbol{f_2})^2 + (\boldsymbol{e_2} \cdot \boldsymbol{f_1})^2 + (\boldsymbol{e_2} \cdot \boldsymbol{f_2})^2]$$

e_1、e_2 and f_1、f_2 are arbitrary orthonormal bases of planes 1 and 2.

A measure of the "angle" between two planes should fulfil the following conditions:

- The value is 1 when the two planes are identical.
- The value is 0 when the two planes are disjoint.



Review of the Measurement Procedure of SOFT

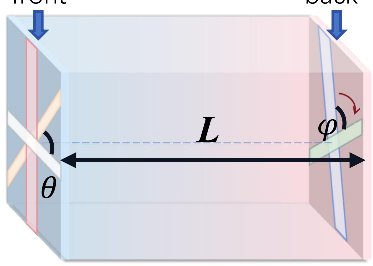
$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Ideal σ Matrix

θ_list: [0°, 60°, 120°]
φ_list: [0°, 60°, 120°]

$$\boldsymbol{U} = M \cdot \boldsymbol{Y}$$

$$\begin{bmatrix} u_{front} \\ u_{back} \\ v_{front} \\ v_{back} \end{bmatrix} = \begin{bmatrix} cos\theta & 0 & -sin\theta & 0 \\ cos\varphi & L \cdot cos\varphi & -sin\varphi & -L \cdot sin\varphi \\ sin\theta & 0 & cos\theta & 0 \\ sin\varphi & L \cdot sin\varphi & cos\varphi & L \cdot cos\varphi \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$



Extract two vectors of the plane and orthonormalize them

Search for the angle pair yielding v = 1 between two sets

Optimal angle measurement sets

$$\sigma = \begin{pmatrix} \langle xx \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle & \langle yx' \rangle & \langle x'y' \rangle \\ \langle xy \rangle & \langle yx' \rangle & \langle yy \rangle & \langle yy' \rangle \\ \langle xy' \rangle & \langle x'y' \rangle & \langle yy' \rangle & \langle y'y' \rangle \end{pmatrix}$$

4D Transverse Phase-Space σ Matrix

Coordinate Transformation

$$\boldsymbol{Y} = P^{-1}\boldsymbol{X} \qquad \boldsymbol{U} = M \cdot \boldsymbol{Y}$$

$$\boldsymbol{U} = M \cdot P^{-1} \cdot \boldsymbol{X}$$

$$\boldsymbol{U} = M_{new} \cdot \boldsymbol{X}$$

Simulation Verification of the Proposed Theory

Two Sets of σ Matrix in Different coupling

$$\sigma_{low} = \begin{bmatrix} 4.79 & -2.54 & -0.14 & -0.17 \\ -2.54 & 2.12 & -0.29 & -0.22 \\ -0.14 & -0.29 & 5.02 & 0.02 \\ -0.17 & -0.22 & 0.02 & 1.20 \end{bmatrix}$$

$$\sigma_{large} = \begin{bmatrix} 8.57 & -4.24 & -3.28 & -1.10 \\ -4.34 & 3.35 & -0.74 & 1.52 \\ -3.28 & -0.74 & 11.20 & -3.05 \\ -1.10 & 1.52 & -3.05 & 1.87 \end{bmatrix}$$

Original Set	Best Pair	
	Low Coupling	Large Coupling
$\theta_1 = 0°, \varphi_1 = 0°$	$\theta_1 = 3°, \varphi_1 = 82°$	$\theta_1 = 19°, \varphi_1 = 5°$
$\theta_1 = 0°, \varphi_1 = 60°$	$\theta_1 = 82°, \varphi_1 = 39°$	$\theta_1 = 0°, \varphi_1 = 178°$
$\theta_1 = 0°, \varphi_1 = 120°$	$\theta_1 = 171°, \varphi_1 = 138°$	$\theta_1 = 166°, \varphi_1 = 176°$
$\theta_1 = 60°, \varphi_1 = 0°$	$\theta_1 = 180°, \varphi_1 = 88°$	$\theta_1 = 1°, \varphi_1 = 144°$
$\theta_1 = 60°, \varphi_1 = 60°$	$\theta_1 = 41°, \varphi_1 = 11°$	$\theta_1 = 177°, \varphi_1 = 139°$
$\theta_1 = 60°, \varphi_1 = 120°$	$\theta_1 = 152°, \varphi_1 = 22°$	$\theta_1 = 177°, \varphi_1 = 177°$
$\theta_1 = 120°, \varphi_1 = 0°$	$\theta_1 = 144°, \varphi_1 = 125°$	$\theta_1 = 6°, \varphi_1 = 170°$
$\theta_1 = 120°, \varphi_1 = 60°$	$\theta_1 = 152°, \varphi_1 = 164°$	$\theta_1 = 174°, \varphi_1 = 168°$
$\theta_1 = 120°, \varphi_1 = 120°$	$\theta_1 = 146°, \varphi_1 = 171°$	$\theta_1 = 145°, \varphi_1 = 177°$

	Low Coupling	*Large Couling*
cond($A_{Initial}$)	*62.781*	*212.901*
cond($A_{Improved}$)	*53.236*	*41.998*
optimized Ratio	*15.20%*	*80.27%*

- Using the 4D geometric tool to select angle pair significantly improves the condition number.
- The improvement is more pronounced under strong coupling conditions.

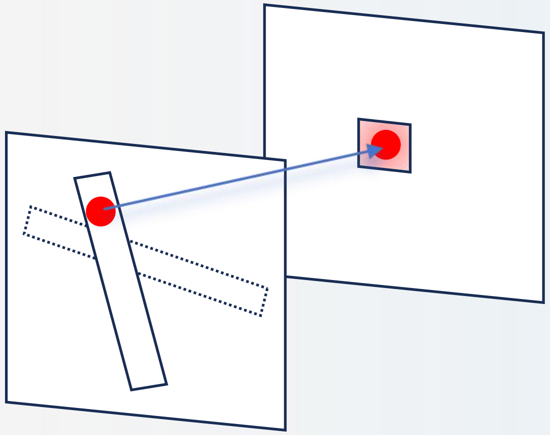
Summary

SOFT offers a robust, compact, and magnet-free solution for 4D phase space measurement.

- The working principle of SOFT has been verified analytically and with simulated tomography experiments.

- Minimizing errors in SOFT measurements requires strategies beyond standard emittance scans. We employed symmetry principles and 4D geometric tools to determine the optimal device and measurement parameters for maximum accuracy.

Final Thoughts



Virtual Pepper Pot & SOFT

Thank you for your attention!

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