Space charge
Touschek lifetime
Intrabeam scattering

Reference

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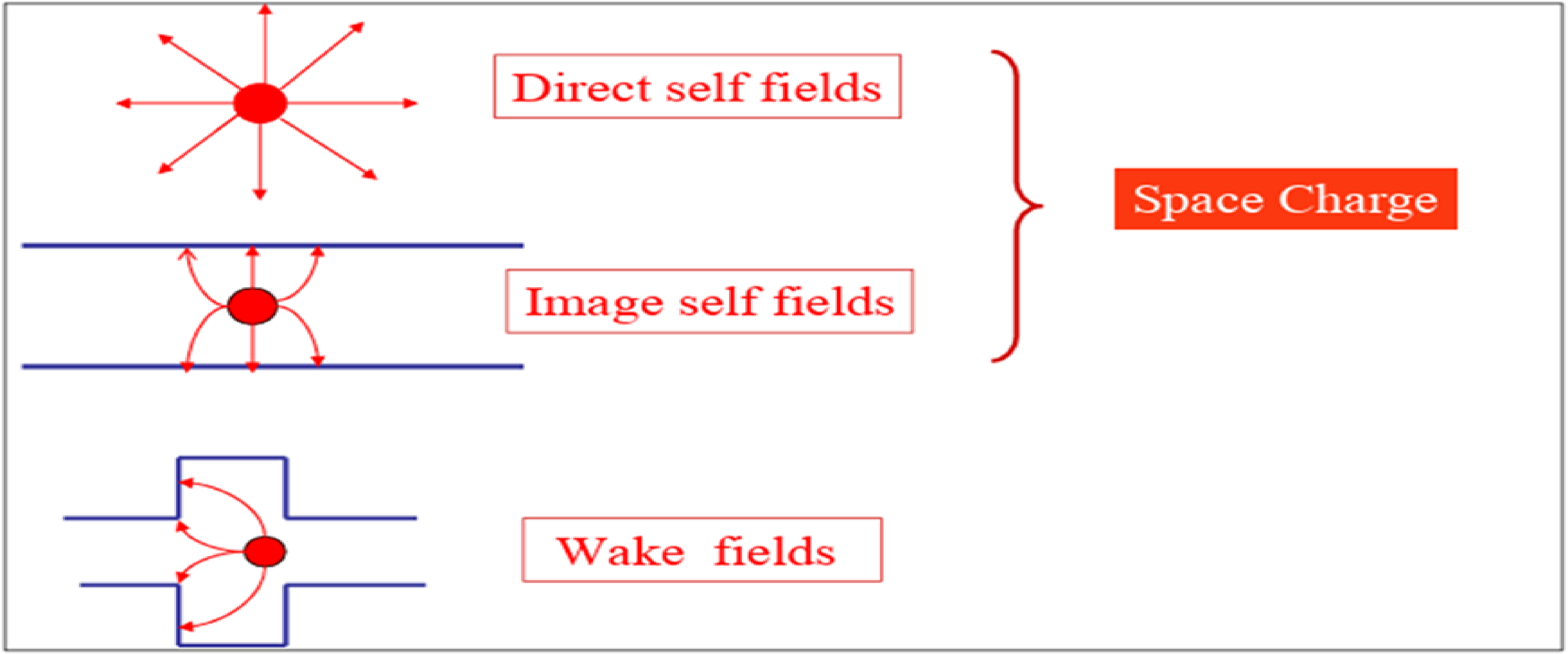
SELF FIELDS AND WAKE FIELDS



Space charge fields

One of the most studied collective effects is the beam's own Coulomb field on a particle in the beam (beam-self interaction). This is called the space charge fields or space charge effect.

Lorentz transformation of electric and magnetic fields

- We consider a charge q which is instantaneously at rest in a frame S'. It experiences an electric field E' and magnetic field B'. We shall assume that the axes of frames are parallel and the origin of S' moves on the x-axis of S with a speed of v. According to S, the charge moves with a speed of v along the x-axis.

$$\vec{F} = q\left(E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}\right) + q\left[v\hat{\imath} \times \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)\right]$$

$$F_x = qE_x \qquad F_y = q\left(E_y - vB_z\right) \qquad F_z = q\left(E_z + vB_y\right)$$

$$F'_x = qE'_x \qquad F'_y = qE'_y \qquad F'_z = qE'_z$$

$$\vec{F}' = q\left(\vec{E}' + \cancelto{0}{\vec{u}'} \times \vec{B}'\right)$$

Lorentz transformation of electric and magnetic fields

- From force transformation equation, where particle is at rest in a frame S',

$$F_x = F'_x \qquad F_y = F'_y/\gamma \qquad F_z = F'_z/\gamma$$

Thus, $$qE_x = qE'_x \qquad q(E_y - vB_z) = qE'_y/\gamma \qquad q(E_z + vB_y) = qE'_z/\gamma$$

Electric field in S'

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

Inverse Transformation,

$$E_x = E'_x \qquad E_y = \gamma(E'_y + vB'_z) \qquad E_z = \gamma(E'_z - vB'_y)$$

Lorentz transformation of electric and magnetic fields

- We now consider q which is moving with a speed u_0' along positive y' axis a frame S'. It experiences an electric field E' and magnetic field B'.

$$F' = q\left(E_x'\hat{\imath} + E_y'\hat{\jmath} + E_z'\hat{k}\right) + q\left[u_0'\hat{\jmath} \times \left(B_x'\hat{\imath} + B_y'\hat{\jmath} + B_z'\hat{k}\right)\right]$$

$$F_x' = q(E_x' + u_0'B_z') \qquad F_y' = qE_y' \qquad F_z' = q(E_z' - u_0'B_x')$$

According to S, the charge moves with a velocity u, the components of which are given by inverse velocity transformations.

$$u_x = \frac{u_x'+v}{1+\frac{vu_x'}{c^2}} = v \qquad u_y = \frac{u_y'}{\gamma\left(1+\frac{vu_x'}{c^2}\right)} = \frac{u_0'}{\gamma}$$

$$u_z = \frac{u_z'}{1+\frac{vu_x'}{c^2}} = 0 \qquad \left(u_x' = 0,\quad u_y' = u_0',\quad u_z = 0\right)$$

Lorentz transformation of electric and magnetic fields

- $\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$

$$= q\left(E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}\right) + q\left[\left(v\hat{\imath} + \frac{u_0'}{\gamma}\hat{\jmath}\right) \times \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)\right]$$

$$= q\left(E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}\right) + q\left[\frac{u_0' B_z}{\gamma}\hat{\imath} - vB_z\hat{\jmath} + \left(vB_y - \frac{u_0' B_x}{\gamma}\right)\hat{k}\right]$$

$$F_x = q\left(E_x + \frac{u_0' B_z}{\gamma}\right) \qquad F_y = q\left(E_y - vB_z\right) \qquad F_z = q\left(E_z + vB_y - \frac{u_0' B_x}{\gamma}\right)$$

$$F_x = \frac{F_x' + vF' \cdot u'/c^2}{1 + vu_x'/c^2} = F_x' + vF_y' u_0'/c^2$$

$$\left(F' \cdot u' = F_x' u_x' + F_y' u_y' + F_z' u_z'\right) \qquad \left(u_y' = u_0'\right)$$

$$F_y = \frac{F_y'}{\gamma\left(1 + vu_x'/c^2\right)} = \frac{F_y'}{\gamma} \qquad \left(u_x' = 0\right)$$

$$F_z = \frac{F_z'}{\gamma\left(1 + vu_x'/c^2\right)} = \frac{F_z'}{\gamma}$$

Lorentz transformation of electric and magnetic fields

- Using the equation corresponding to x-component,

$$q(E_x + u_0' B_z/\gamma) = q(E_x' + u_0' B_z') + \frac{v}{c^2} q E_y' u_0'$$

Using the transformation $E_x' = E_x$

$$(E_x + u_0' B_z/\gamma) = (E_x' + u_0' B_z') + \frac{v}{c^2} E_y' u_0'$$

$$\rightarrow \boxed{B_z = \gamma\left(B_z' + \frac{v}{c^2} E_y'\right)}$$

Inverse transformation of z-component of magnetic field

Using the equation corresponding to z-component

$$\gamma q\left(E_z + vB_y - u_0' B_x/\gamma\right) = q(E_z' - u_0' B_x')$$

Using the transformation $E_z' = \gamma\left(E_z + vB_y\right)$

$$\gamma\left(E_z + vB_y - u_0' B_x/\gamma\right) = \gamma\left(E_z + vB_y\right) - u_0' B_x'$$

$$\rightarrow \boxed{B_x = B_x'}$$

Inverse transformation of x-component of magnetic field

Lorentz transformation of electric and magnetic fields

- For obtaining the transformation of y-component of field, we assume that the charge is moving along z' direction of frame S'.

$$F' = q\left(E'_x\hat{\imath} + E'_y\hat{\jmath} + E'_z\hat{k}\right) + q\left[u'_0\hat{k} \times \left(B'_x\hat{\imath} + B'_y\hat{\jmath} + B'_z\hat{k}\right)\right]$$

$$F'_x = q\left(E'_x - u'_0 B'_y\right) \qquad F'_y = q\left(E'_y + u'_0 B'_0\right) \qquad F'_z = qE'_z$$

$$F = q\left(E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}\right) + q\left[\left(v\hat{\imath} + \frac{u'_0}{\gamma}\hat{k}\right) \times \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)\right]$$

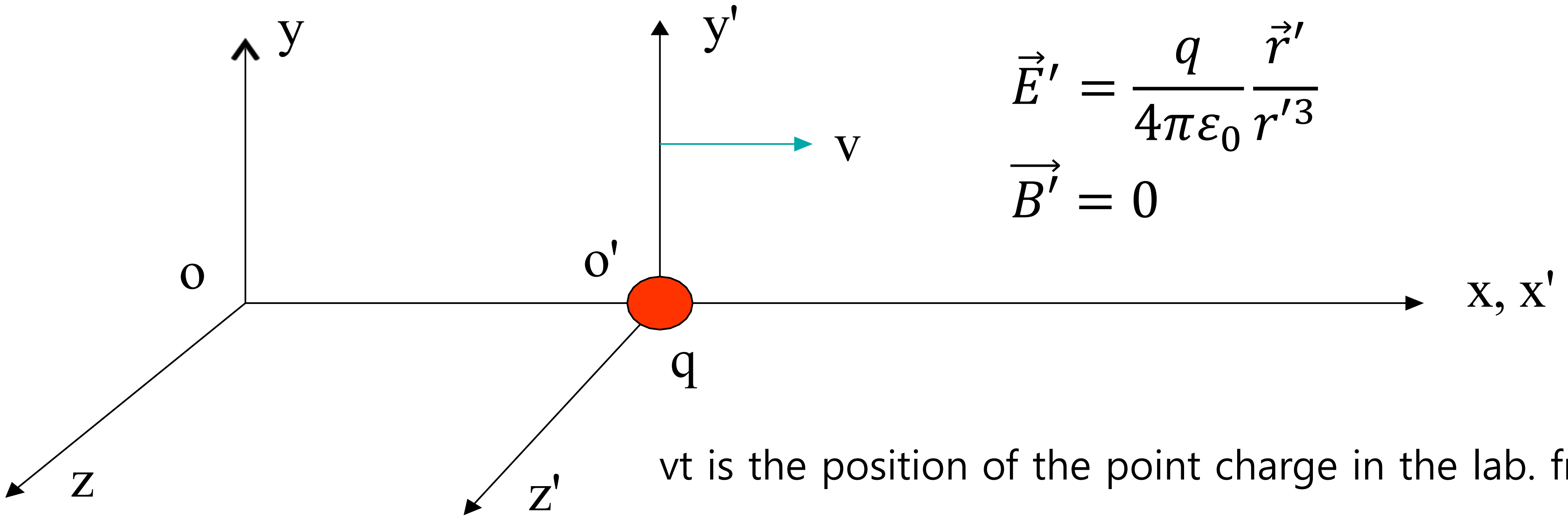
$$F_x = q\left(E_x - \frac{u'_0}{\gamma}B_y\right) \qquad F_y = q\left(E_y - vB_z + \frac{u'_0}{\gamma}B_x\right) \qquad F_z = q\left(E_z + vB_y\right)$$

From $F_x = F'_x + \frac{vF'_z u'_0}{c^2}$, $q\left(E_x - \frac{u'_0}{\gamma}B_y\right) = q\left(E'_x - u'_0 B'_y\right) + \frac{v}{c^2}qE'_z u'_0$

$$E_x = E'_x \qquad \boxed{B_y = \gamma\left(B'_y - \frac{v}{c^2}E'_z\right)}$$

- $\boxed{B'_x = B_x \quad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \quad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)}$

Fields of a point charge with uniform motion



$$\vec{E}' = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}'}{r'^3}$$

$$\vec{B'} = 0$$

vt is the position of the point charge in the lab. frame O.

- In the moving frame O', the charge is at rest
- Electric field is radial with spherical symmetry
- Magnetic field is zero

$$E'_x = \frac{q}{4\pi\varepsilon_0}\frac{x'}{r'^3} \qquad E'_y = \frac{q}{4\pi\varepsilon_0}\frac{y'}{r'^3} \qquad E'_z = \frac{q}{4\pi\varepsilon_0}\frac{z'}{r'^3}$$

Relativistic transforms of the fields from O′ to O

$$\begin{cases} E_x = E'_x \\ E_y = \gamma\left(E'_y + vB'_z\right) \\ E_z = \gamma\left(E'_z - vB'_y\right) \end{cases}$$

$$\begin{cases} B_x = B'_x \\ B_y = \gamma\left(B'_y - vE'_z/c^2\right) \\ B_z = \gamma\left(B'_z + vE'_y/c^2\right) \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ ct' = \gamma\left(ct - \frac{v}{c}x\right) \end{cases}$$

$$r' = \sqrt{x'^2 + y'^2 + z'^2}$$

$$r' = \sqrt{\gamma^2(x - vt)^2 + y^2 + z^2}$$

$$B' = 0$$

Relativistic transforms of the fields from O′ to O

$$E_x = E'_x = \frac{q}{4\pi\varepsilon_0}\frac{x'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(x-vt)}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\varepsilon_0}\frac{y'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

$$E_z = \gamma E'_z = \frac{q}{4\pi\varepsilon_0}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma z}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

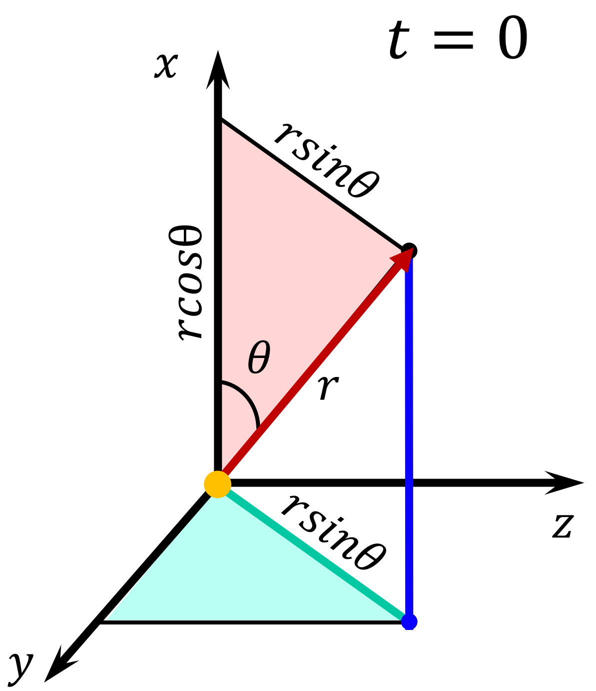
Notice the factor γ that appears in x-dependence of the fields. With increasing velocity, the fields become “flattened” towards the plane perpendicular to the direction of motion of the charge.

The field pattern is moving with the charge and it can be observed at $t=0$:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\vec{r}}{[\gamma^2x^2+y^2+z^2]^{3/2}}$$

The fields have lost the spherical symmetry.

Relativistic transforms of the fields from O′ to O



$x = r\cos\theta$

$y^2 + z^2 = r^2 \sin^2\theta$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\gamma \vec{r}}{[\gamma^2 x^2 + y^2 + z^2]^{3/2}}$$

$$\gamma^2 x^2 + y^2 + z^2 = r^2 \gamma^2 \left(1 - \beta^2 \sin^2\theta\right)$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\left(1 - \beta^2\right)}{r^2 (1 - \beta^2 \sin^2\theta)^{3/2}} \frac{\vec{r}}{r}$$

Notice the factor γ that appears in x-dependence of the fields. With increasing velocity, the fields become "flattened" towards the plane perpendicular to the direction of motion of the charge.

Relativistic transforms of the fields from O′ to O

- Consider the fields along the axes for the case t=0

$$E_x(y = z = 0) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\gamma^2 x^2}$$

$$E_y(x = z = 0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{y^2}$$

$$E_z(x = y = 0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{z^2}$$

$$B_x = 0$$

$$B_y(x = y = 0) = \frac{-v\,q}{c^2\,4\pi\varepsilon_0}\frac{\gamma}{z^2}$$

$$B_z(x = z = 0) = \frac{v\,q}{c^2\,4\pi\varepsilon_0}\frac{\gamma}{y^2}$$

Relativistic transforms of the fields from O′ to O

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

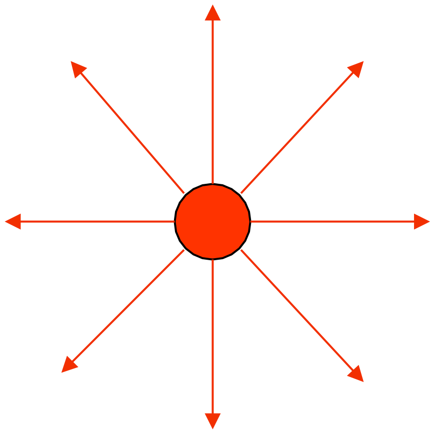
$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{(1-\beta^2)}{r^2(1-\beta^2\sin^2\theta)^{3/2}}\frac{\vec{r}}{r}$$

$$\beta = 0 \Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\vec{r}}{r}$$

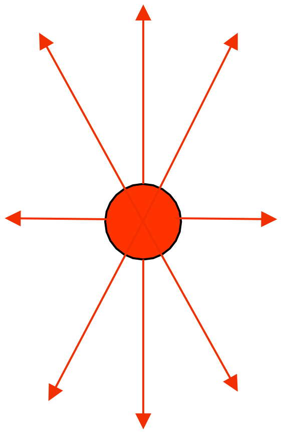
$$\theta = 0 \Rightarrow E_{\parallel} = \lim_{\gamma\to\infty}\frac{q}{4\pi\varepsilon_0}\frac{1}{\gamma^2 r^2}\frac{\vec{r}}{r} = 0$$

$$\theta = \pi/2 \Rightarrow E_{\perp} = \lim_{\gamma\to\infty}\frac{q}{4\pi\varepsilon_0}\frac{\gamma}{r^2}\frac{\vec{r}}{r} = \infty$$

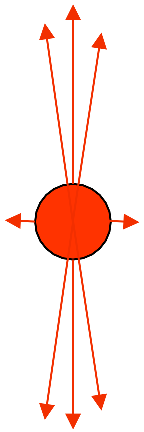
$\gamma = 1$



$\gamma > 1$



$\gamma >> 1$



B is transverse to direction of motion

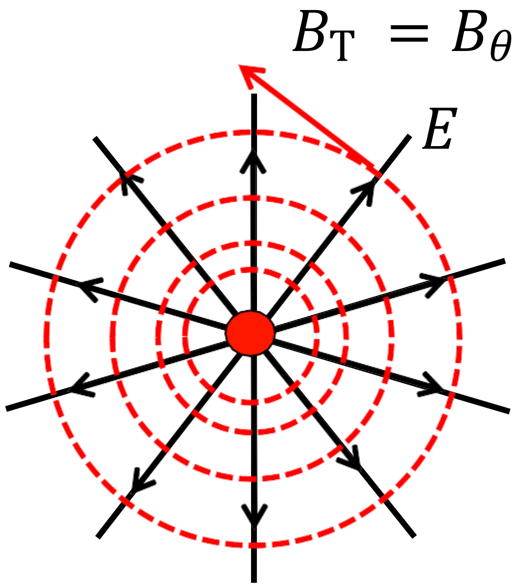
$$B' = 0$$

$$\begin{cases} B_x = B'_x \\ B_y = \gamma\left(B'_y - vE'_z/c^2\right) \\ B_z = \gamma\left(B'_z + vE'_y/c^2\right) \end{cases} \Rightarrow \quad \begin{aligned} B_x &= 0 \\ B_y &= -vE_z/c^2 \\ B_z &= vE_y/c^2 \end{aligned}$$

$$\vec{B}_\perp = \frac{\vec{v} \times \vec{E}}{c^2}$$

Magnetic field is "flattened" at high velocities, in the same way as electric field.

$$B_\perp = B_\theta = \frac{vE_r}{c^2} = \frac{\beta E_r}{c}$$



$$\gamma \to \infty$$

Electric and magnetic fields around a relativistic charged particle are "flattened" toward a plane perpendicular to the direction of motion of charged particle.

Direct space charge

Self-fields

Unbunched beam of circular cross section (radius a) and uniform charge density moves with constant velocity $v = \beta c$. It has a line charge density of $\lambda = \pi a^2 \rho$, current density [A/m] of $J = \beta c \rho$, and total current of $I = \beta c \lambda$. Using polar coordinates (r, ϕ), due to symmetry, electric field has a radial component (E_r) and magnetic field lines are circles around the cylinder (B_ϕ component only)

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\iiint \left(\vec{\nabla} \cdot \vec{E}\right) dV = \iint \vec{E}\, d\vec{S}$$

$$\frac{\rho}{\epsilon_0} \pi l r^2 = E_r 2\pi l r$$

$$\boxed{E_r = \frac{\rho r}{2\epsilon_0} = \frac{I}{2\pi\epsilon_0 \beta c} \frac{r}{a^2}}$$

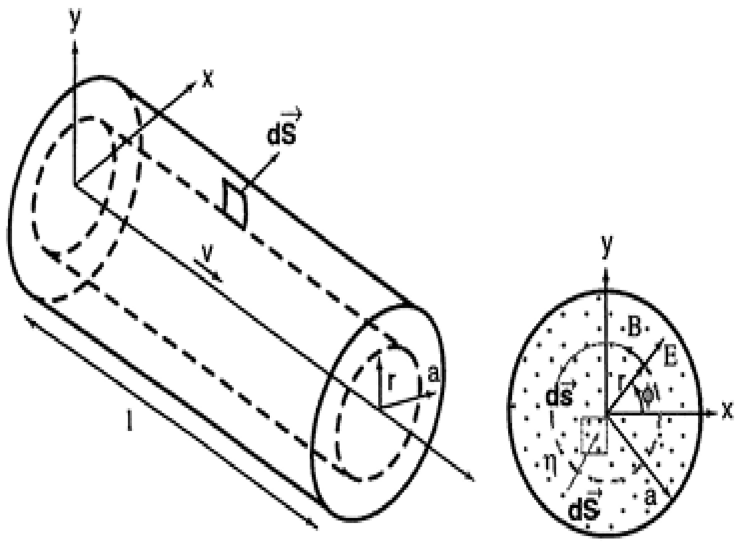
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\oint \vec{B} d\vec{s} = \iint \left(\vec{\nabla} \times \vec{B}\right) d\vec{S}$$

$$B_\phi 2\pi r = \mu_0 \beta c \rho \pi r^2$$

$$\boxed{B_\phi = \frac{I}{2\pi\epsilon_0 c^2} \frac{r}{a^2}}$$

$$E_r = \frac{c}{\beta} B_\phi$$



Both electric and magnetic fields vanish at r =0 and increase linearly with r.

Direct space charge Force

These fields exert a force on a test particles at radius r

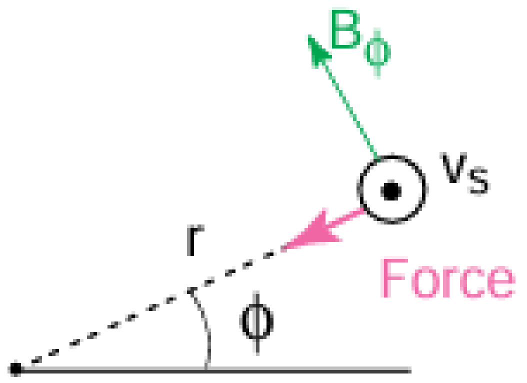
$$\vec{F} = e(\vec{E} + [\vec{v} \times \vec{B}])$$

Force vector has a purely radial component.

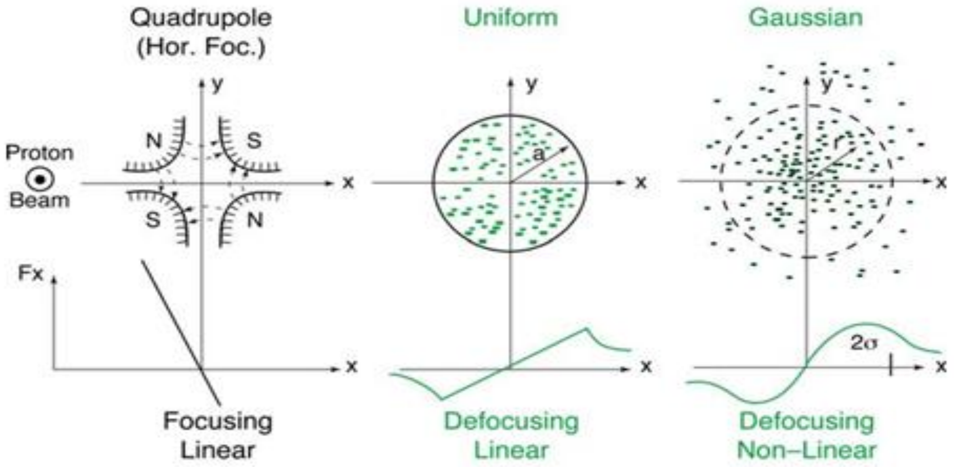
$$F_r = e\left(E_r - v_s B_\phi\right)$$

$$F_r = \frac{eI}{2\pi\epsilon_0 \beta c}\left(1-\beta^2\right)\frac{r}{a^2} = \frac{eI}{2\pi\epsilon_0 \beta c}\frac{1}{\gamma^2}\frac{r}{a^2}$$

$$F_x = \frac{eI}{2\pi\epsilon_0 c\beta\gamma^2 a^2}x, \qquad F_y = \frac{eI}{2\pi\epsilon_0 c\beta\gamma^2 a^2}y$$



Magnetic force on test particle at r



While quadrupole is focusing in one and defocusing in the other plane, direct space charge leads to defocusing in both planes.

Beam transport with space charge

$$x'' + K(s)x = 0$$

$$x'' + \big(K(s) + K_{SC}(s)\big)x = 0$$

$K_{SC}(s)$ describes defocusing action of space charge.

$$x'' = \frac{d^2x}{ds^2} = \frac{1}{\beta^2c^2}\frac{d^2x}{dt^2} = \frac{\ddot{x}}{\beta^2c^2} = \frac{1}{\beta^2c^2}\frac{F_x}{m_0\gamma} = \frac{2r_0I}{ea^2\beta^3\gamma^3c}x,$$

$$r_0 = e^2/(4\pi\epsilon_0 m_0 c^2)$$

$$x'' + \left(K(s) - \frac{2r_0I}{ea^2\beta^3\gamma^3c}\right)x = 0$$

Hill's equation with space charge

negative sign of space charge term, reducing overall focusing.

Direct space charge in a synchrotron : Incoherent tune shift

Direct space charge leads to defocusing in either plane.

Beam will experience a lowering of their betatron tunes by

$$\Delta Q_x = \frac{1}{4\pi}\int_0^{2\pi R} K_x(s)\beta_x(s)ds = \frac{1}{4\pi}\int_0^{2\pi R} K_{SC}(s)\beta_x(s)ds$$

$$\Delta Q_x = -\frac{1}{4\pi}\int_0^{2\pi R} \frac{2r_0 I}{e\beta^3\gamma^3 c}\frac{\beta_x(s)}{a^2}ds = -\frac{r_0 R I}{e\beta^3\gamma^3 c}\left\langle\frac{\beta_x(s)}{a^2(s)}\right\rangle, \qquad (I = \frac{Ne\beta c}{2\pi R})$$

$$\Delta Q_{x,y} = -\frac{r_0 N}{2\pi\varepsilon_{x,y}\beta^2\gamma^3} \qquad (a^2{}_{x,y}(s) = \beta_{x,y}(s)\,\varepsilon_{x,y})$$

It scales with $1/\gamma^3$

Direct space charge for a non-uniform beam

For a non-uniform distribution, bi-Gaussian density in the circular beam cross section

$$\rho(r) = \frac{I}{2\pi\beta c\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, \qquad r = \sqrt{x^2 + y^2}, \qquad E_r = \frac{I}{2\pi\epsilon_0 \beta c}\frac{1}{r}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right), \qquad B_\phi = \frac{I}{2\pi\epsilon_0 c^2}\frac{1}{r}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)$$

$$F_r(r) = \frac{eI}{2\pi\epsilon_0\beta c\gamma^2}\frac{1}{r}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)$$

It is no longer linear in r, thus the defocusing becomes betatron-amplitude dependent

For small r (near the beam center)

$$F_r(r) = \frac{eI}{2\pi\epsilon_0\beta\gamma^2 c}\frac{1}{r}\left(1 - 1 + \frac{r^2}{2\sigma^2} - \cdots\right) \approx \frac{eI}{2\pi\epsilon_0\beta\gamma^2 c}\frac{r}{2\sigma^2}$$

$$F_y \approx \frac{eI}{2\pi\epsilon_0\beta\gamma^2 c}\frac{y}{2\sigma^2}$$

Resulting in small-amplitude (vertical) tune shift of

$$\Delta Q_y = -\frac{r_0 I R}{e c\beta^3\gamma^3}\left\langle\frac{\beta_y}{2\sigma^2}\right\rangle = -\frac{r_0 N}{2\pi\beta^2\gamma^3}\frac{2}{\varepsilon_y} \quad for\ r \ll \sigma$$

95% emittance $\varepsilon_y = 4\sigma^2/\beta_y$

It is twice compared to a uniform beam of the same cross-sectional size and intensity.

The envelope equation

- Since we know the equation of motion for individual particles, we can work out the equation of motion for the beam distribution. The equation of motion for the rms beam size is known as the envelope equation:

$$\frac{d^2\sigma_x}{ds^2} + k_1\sigma_x - \frac{\varepsilon^2_x}{\sigma^3_x} - \frac{K}{2(\sigma_x+\sigma_y)} = 0$$

k_1 is the quadrupole focusing strength, σ_x is rms horizontal beam size: $\sigma_x = \sqrt{\langle x^2 \rangle}$, and emittance ε_x is given by:

$$\epsilon_x = \sqrt{\langle x^2\rangle\langle x'^2 - \langle xx'\rangle^2}.$$

Similar equations apply for the vertical motion.

$$\frac{d^2\sigma_y}{ds^2} + k_1\sigma_y - \frac{\varepsilon^2_y}{\sigma^3_y} - \frac{K}{2(\sigma_x+\sigma_y)} = 0$$

Envelope equation: continuous beam with elliptical symmetry

- Each term in the envelope equation has a clear physical origin

$$\frac{d^2\sigma_x}{ds^2} + k_1\sigma_x - \frac{\varepsilon^2{}_x}{\sigma^3{}_x} - \frac{K}{2(\sigma_x+\sigma_y)} = 0$$

  - $k_1\sigma_x$ represents the quadrupole focusing
  - $\varepsilon^2{}_x/\sigma^3{}_x$ represents the evolution of the beam size arising from the beam emittance (non-zero divergence)
  - $K/2(\sigma_x + \sigma_y)$ represents the defocusing effect of the space-charge forces

If the emittance term is much larger than the space-charge term, then the beam transport is said to be emittance dominated.

If the space-charge term is much larger than the emittance term, then the beam transport is said to be space-charge dominated.

Space-charge tune shifts

- The defocusing effects of space-charge forces lead to changes in the betatron oscillation frequencies.
- For a continuous beam with uniform charge density, all particles experience the same frequency shifts.
- For a bunched beam the situation is more complicated: the space-charge forces are nonlinear, and depend on the longitudinal position of the particle in the bunch.

Incoherent space-charge tune shifts

- Consider the particles in a bunch in a storage ring. Although each particle within the bunch experiences a betatron frequency shift from space-charge forces, if we neglect interactions with the vacuum chamber, then the overall space-charge force on the bunch is zero.

- This kind of tune shift is sometimes called an incoherent tune shift: it cannot be measured by observing the coherent motion of a bunch of particles.

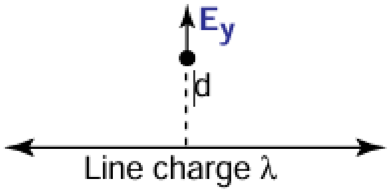
- For particles in a bunch with non-uniform density, there is a tune spread representing the range of tune shifts for different particles. The vertical tune spread can be estimated using the formula:

$$\Delta\nu_y = -\frac{K}{4\pi} \oint \frac{ds\,\sigma_y}{\sigma_y\,(\sigma_x + \sigma_y)}$$

TUNE SHIFT WITH WALL EFFECTS

- 'Incoherent' tune shift due to conductive walls

$$E_y = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{d}$$



Beam, whose barycenter is halfway between two parallel conducting plates, gives rise to an infinite number of image line charges of alternating sign at positions 2h, 4h, 6h,....–2h ,–4h,....

First pair of image line charges, positioned at 2h, –2h, yields the vertical field at point y inside the beam generated by two images

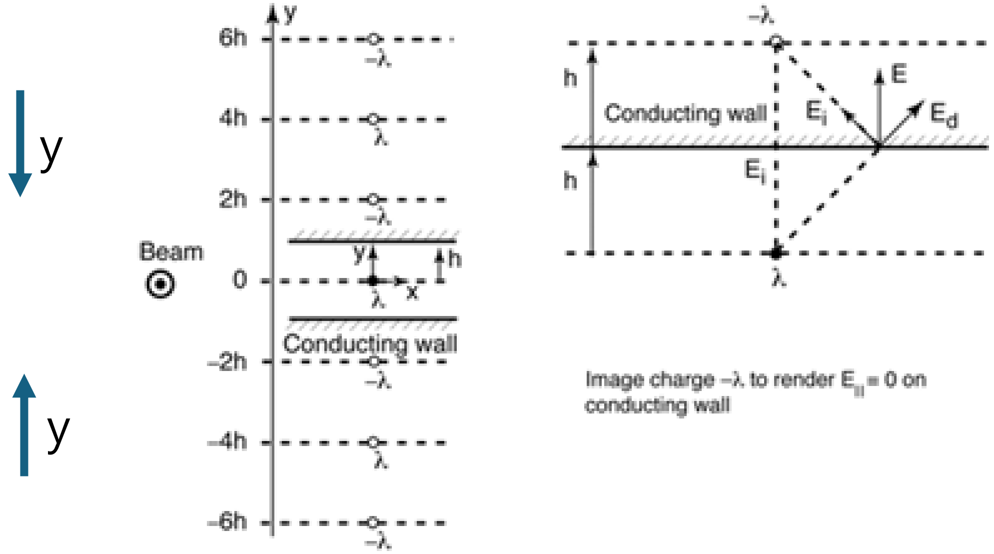
$$E_{i1y} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{2h-y} - \frac{1}{2h+y}\right)$$

n^{th} pair of line charges, positioned at 2nh, –2nh

$$E_{iny} = (-1)^n\frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{2nh+y} - \frac{1}{2nh-y}\right) = (-1)^n\frac{\lambda}{4\pi\epsilon_0}\frac{y}{n^2h^2}$$

$$E_{iy} = \sum_1^\infty E_{iny} = \frac{\lambda}{4\pi\epsilon_0 h^2}\sum_1^\infty\frac{(-1)^n}{n^2}\,y = \frac{\lambda}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12}y,$$

$$\sum_1^\infty\frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$$



TUNE SHIFT WITH WALL EFFECTS

Incoherent tune shift due to conductive walls

Between the parallel plates, there are no image charges, therefore $\vec{\nabla} \cdot \vec{E} = 0$

$$\nabla \cdot \vec{E}_i = \frac{\partial E_{ix}}{\partial x} + \frac{\partial E_{iy}}{\partial y} = 0 \quad \Rightarrow \quad E_{ix} = -\frac{\lambda}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12}x$$

$$F_{iy} = \frac{e\lambda}{\pi\epsilon_0 h^2}\frac{\pi^2}{48}y, \qquad F_{ix} = -\frac{e\lambda}{\pi\epsilon_0 h^2}\frac{\pi^2}{48}x$$

Total incoherent tune shift of a round beam between parallel conducting walls

$$\Delta Q_x = -\frac{2r_0 IR\langle\beta_x\rangle}{ec\beta^3\gamma}\left(\frac{1}{2\langle a^2\rangle\gamma^2} - \frac{\pi^2}{48h^2}\right) \qquad \text{(direct – image)}$$

$$\Delta Q_y = -\frac{2r_0 IR\langle\beta_y\rangle}{ec\beta^3\gamma}\left(\frac{1}{2\langle a^2\rangle\gamma^2} + \frac{\pi^2}{48h^2}\right) \qquad \text{(direct – image)}$$

Electric image field is vertically defocusing, but horizontally focusing , the field is larger for small chamber height h.
Image effects decrease with $1/\gamma$, much slower than the direct space-charge term ($1/\gamma^3$), and thus are of some concern for electron and high-energy proton machines.

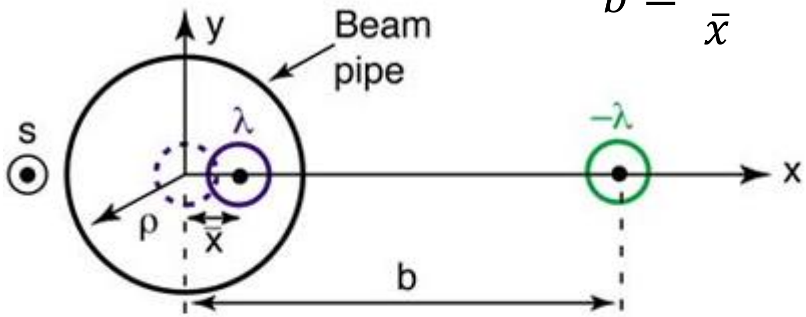
TUNE SHIFT WITH WALL EFFECTS

coherent tune shift due to conductive walls

- a beam with line charge and radius performing coherent oscillations of its center of mass $\bar{x}$ inside the round beam pipe with radius ρ ($a \ll$ρ) . The displaced line charge λ induces surface charges on the inside of the beam pipe which can be represented by an image line charge - λ at distance b.

$$b = \frac{\rho^2}{\bar{x}}$$

Image charge pulls the beam away from the center of the beam pipe: its effect is defocusing.



$$E_{ix}(\bar{x}) = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{b-\bar{x}} \approx \frac{\lambda}{2\pi\epsilon_0}\frac{1}{b} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{\rho^2}\bar{x}$$

$$F_{ix}(\bar{x}) = \frac{e\lambda}{2\pi\epsilon_0}\frac{1}{\rho^2}\,\bar{x}$$

For symmetry reasons, vertical field and force are the same as the horizontal ones,

Coherent tune shift

$$\Delta Q_{x,y,coh} = -\frac{r_0 R\langle\beta_{x,y}\rangle I}{ec\beta^3\gamma\rho^2} = -\frac{r_0\langle\beta_{x,y}\rangle}{2\pi\beta^2}\frac{N}{\gamma\rho^2}$$

Laslett tune shifts

- In addition to direct forces between particles in a bunch, particles experience forces from image charges in the walls of the vacuum chamber.

- The forces from image charges depend on the geometry of the vacuum chamber, and the position of the bunch within the chamber.

- Tune shifts resulting from image charges are known as Laslett tune shifts

- In general, there will be coherent Laslett tune shifts as well as incoherent Laslett tune shifts

Touschek and intrabeam scattering

Both Touschek effect and intrabeam scattering originate from electron-electron scattering in a bunch. Scattering produces changes in the momenta of the electrons.

- Particle longitudinal momentum outside momentum aperture $\gamma\sigma_{x'} > \varepsilon_{acc}$: the particles are lost. Touschek effect : one of the major particle loss mechanism in synchrotron light sources → Touschek lifetime
- Particle longitudinal momentum inside the momentum aperture $\gamma\sigma_{x'} > \varepsilon_{acc}$: intrabeam scattering excites particle oscillations: beam dimensions are increased and energy spread increases → IBS growth rates and equilibrium emittances

- These effects are a source of concern for the operation of ultra low emittance rings -> **can limit the performance of low emittance rings**

TOUSCHEK EFFECT

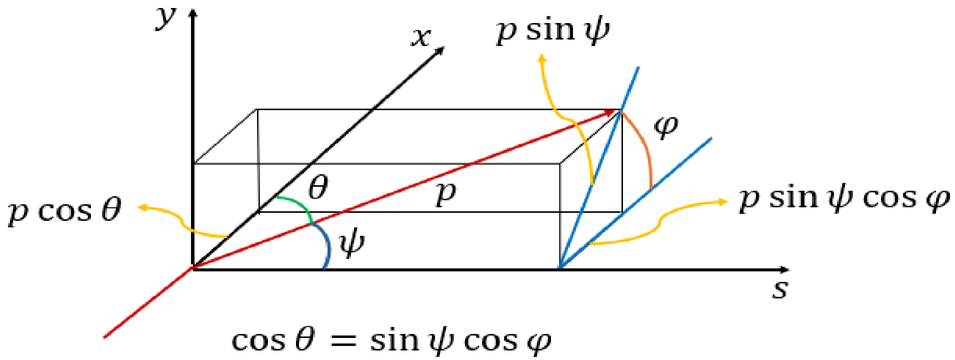
- Coulomb scattering of charged particles traveling together causes an exchange of momentum between the transverse and longitudinal directions.
- Due to relativistic effects, the momentum transferred from the transverse to the longitudinal direction is enhanced by the factor γ.
- For stored beam, particles are lost if their longitudinal momentum deviation exceeds the rf bucket or the momentum aperture determined by the lattice.
-
- A particle with horizontal betatron amplitude $\bar{x}$ have a maximum horizontal velocity $x' = \dfrac{\bar{x}}{\beta_x} = \dfrac{\bar{x}}{\lambda_x} = \dfrac{px}{p}$

  ($x = \bar{x}\sin(s/\beta_x) = \bar{x}\sin(s/\lambda_x)$, λ_x betatron wavelength)
- It corresponds to a transverse momentum $p_x = p\,\bar{x}/\beta_x$
- Consider machine with $\bar{x} = 10^{-4}$ m, $\beta_x = 10$ m, E=4 GeV. Transverse momentum is 40 keV. When transferred into longitudinal direction, it becomes $\Delta E = \gamma p_x = 313$MeV, $\Delta E/E = 7.8\%$ that is larger than energy acceptance

TOUSCHEK EFFECT

The effect can be investigated in CM system where the particles are non-relativistic.
Moller differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{4r_0^2}{\beta^4}\left[\frac{4}{sin^4\theta} - \frac{3}{sin^2\theta}\right]$$



$\cos\theta = \sin\psi\cos\varphi$

$d\Omega = \sin\psi\, d\psi d\varphi$

-In CMS, longitudinal component of the momentum due to scattering is $p_s = p\cos\psi$

-In Lab. system it becomes $p'_s = \gamma\,(p_s - \frac{\beta}{c}E) \sim \gamma p_s = \gamma\ p\cos\psi$ (particles are nonrelativistic in CMS)
Momentum transfer in Lab. System is amplified by a factor by γ.
If $\gamma\ p_s$ is larger than momentum acceptance Δp_A, scattered particles are lost.

-Condition for losing particle is $|\cos\psi| > \frac{\Delta P_A}{\gamma p} = \mu$

$$\sigma = \int_{|\cos\psi \geq \mu|} d\sigma$$

TOUSCHEK EFFECT

- $\sigma_T = \frac{4r_0^2}{(v/c)^4} \int_0^{\cos^{-1}\mu} \sin\psi \, d\psi \cdot 2\int_0^{\pi} d\phi \left[\frac{4}{(1-\sin^2\psi\cos^2\phi)^2} - \frac{3}{(1-\sin^2\psi\cos^2\phi)}\right]$

$$= \frac{8\pi r_0^2}{(v/c)^4}\left[\frac{1}{\mu^2} - 1 + \ln\mu\right] \qquad (\mu = \cos\psi)$$

$$\boxed{\frac{1}{\tau} = \frac{N r_0^2 c}{8\pi\sigma_x\sigma_y\sigma_s}\frac{1}{\gamma^2\varepsilon_{acc}^3}D(\xi)}$$

$$D(\xi) = \sqrt{\xi}\left[-\frac{3}{2}e^{-\xi} + \frac{\xi}{2}\int_\xi^\infty \frac{\ln u}{u}e^{-u}du + \frac{1}{2}(3\xi - \xi\ln\xi + 2)\int_\xi^\infty \frac{e^{-u}}{u}du\right]$$

$\varepsilon_{acc} = dp / p_0$: momentum accep

$$\xi = [\varepsilon_{acc} / \gamma\sigma'_x]^2 \approx \frac{\varepsilon_{acc}^2}{\gamma^2}\frac{\beta_x}{\varepsilon_x}$$

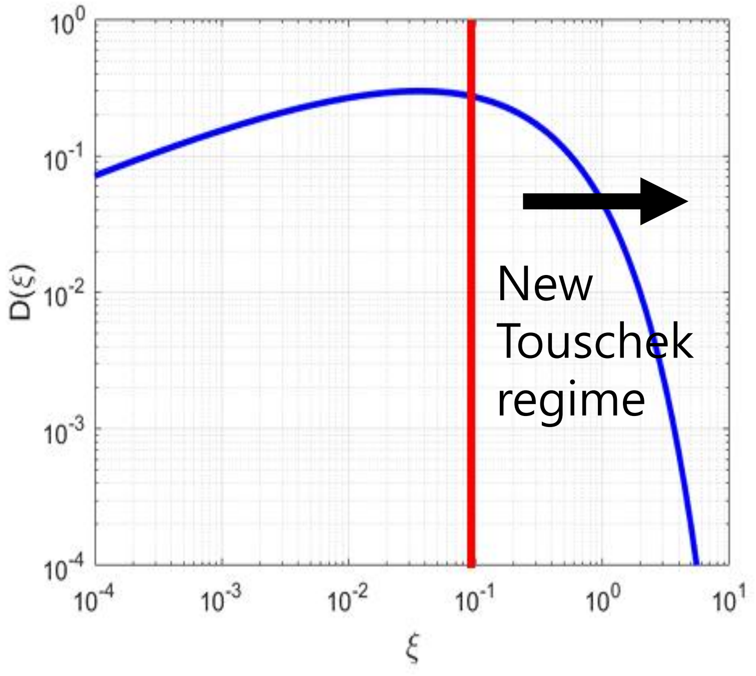
Touschek lifetime of 3rd generation light sources decrease with the emittance as a consequence of the decrease bunch volume.

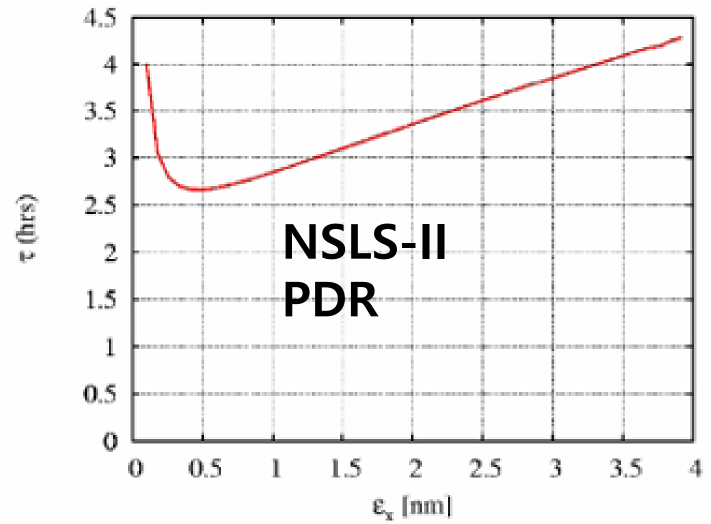
Touschek lifetime is increased by
- increasing momentum aperture
- bunch lengthening cavities
- operation with large coupling
- high energy rings are favoured
- lower charge

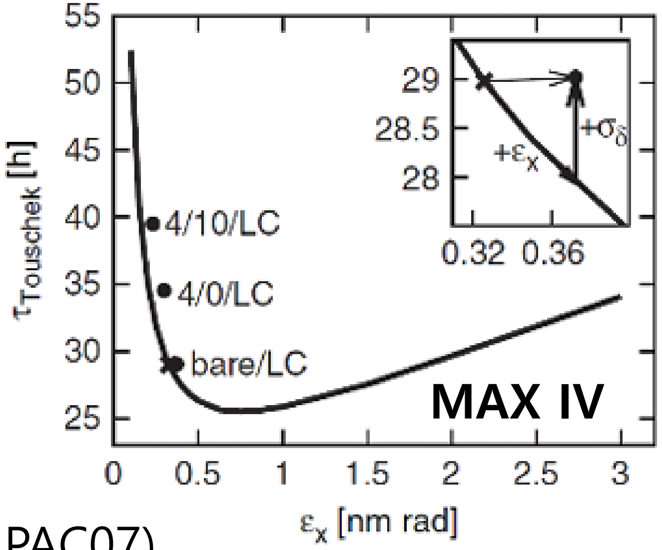
Touschek lifetime

3rd generation light sources operate in the region where D(ξ) is flat
(ε_{acc} = 3%, beta 10 m, 3 GeV, ε_x = 3 nm) : ξ~ 0.09

S. C. Leemann et al., PRST-AB 12, 120701 (2009)







B. Podobedov, in LER14 (and PAC07)

Lifetime increase for small emittance is clearly visible.

lower emittance → small transverse momentum $\sigma_{x'} \rightarrow \gamma\sigma_{x'} < \varepsilon_{acc}$

New upgrades will probe the new Touschek regime
MAX-IV –like (εacc= 4%; beta 5 m; 3 GeV; εx= 0.3 nm) ξ~ 0.77
PETRA IV –like (εacc= 3%; beta 5 m; 6 GeV; εx= 20 pm) ξ~ 1.6

NSLS-II emittance (2 nm bare - 0.9 nm) still not in new Touschek regime mode.
MAX IV emittance (330 pm bare - 200 pm) at the threshold of the new Touschek lifetime regime.

Landau Cavities for Touschek lifetime

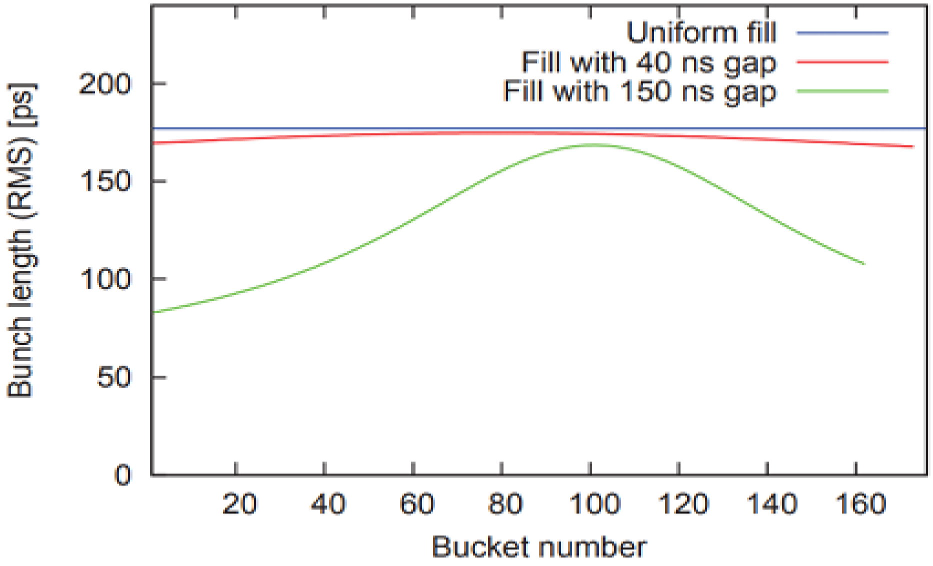
- Harmonic cavities are used to lengthen the bunch and have been used in many light sources (SLS, ELETTRA, BESSY, ALS, MAX IV, …).

- Decreases charge density → bunch lengthening factors ~5 at MAX IV
  - Touschek lifetime increase by the same factor.

- transient beam loading, fill pattern dependence and current dependence

- Landau cavities are effective tool to reduce the emittance growth due to collective effects.

An essential tool in low emittance lattices



Transient beam loading evaluated at MAX IV with a 5 HC. Without HC the bunch length is 34 ps

PIWINSKI'S FORMULA

- Touschek lifetime can be defined by $\frac{1}{T_i} = < \frac{R}{N_o} >$

  N_o is number of particles in a bunch. R is the total number of scattering events per unit time.

$$R = \frac{r_p^2 c \beta_x \beta_y \sigma_h N_p^2}{8\sqrt{\pi}\beta^2\gamma^4\sigma_{x\beta}^2\sigma_{y\beta}^2\sigma_s\sigma_p} F(\tau_m, B_1, B_2), \quad (5)$$

with

$$B_1 = \frac{1}{2\beta^2\gamma^2}\left[\frac{\beta_x^2}{\sigma_{x\beta}^2} - \frac{\beta_x^2\sigma_h^2\tilde{D}_x^2}{\sigma_{x\beta}^4} + \frac{\beta_y^2}{\sigma_{y\beta}^2} - \frac{\beta_y^2\sigma_h^2\tilde{D}_y^2}{\sigma_{y\beta}^4}\right], \quad (6)$$

$$B_2^2 = B_1^2 - \frac{\beta_x^2\beta_y^2\sigma_h^2}{\beta^4\gamma^4\sigma_{x\beta}^4\sigma_{y\beta}^4\sigma_p^2}(\sigma_x^2\sigma_y^2 - \sigma_p^4 D_x^2 D_y^2), \quad (7)$$

$$\tau_m = \beta^2\delta_m^2 = \beta^2\left(\frac{\Delta p_m}{p}\right)^2, \quad (8)$$

$$\sigma_h = \frac{\sigma_{x\beta}\sigma_{y\beta}\sigma_p}{\sqrt{\tilde{\sigma}_x^2\sigma_{y\beta}^2 + \tilde{\sigma}_y^2\sigma_{x\beta}^2 - \sigma_{x\beta}^2\sigma_{y\beta}^2}}, \quad (9)$$

$$\tilde{D}_{x,y} = \alpha_{x,y}D_{x,y} + \beta_{x,y}D'_{x,y}, \quad (10)$$

$$\tilde{\sigma}_{x,y}^2 = \sigma_{x\beta,y\beta}^2 + \sigma_p^2(D_{x,y}^2 + \tilde{D}_{x,y}^2), \quad (11)$$

$$F = \int_{\tau_m}^{\infty} e^{-B_1\tau} I_0(B_2\tau)\frac{\sqrt{\tau}d\tau}{\sqrt{1+\tau}}((2+\frac{1}{\tau})^2(\frac{\tau/\tau_m}{1+\tau} - 1) + 1 - \frac{\sqrt{1+\tau}}{\sqrt{\tau/\tau_m}} - \frac{4\tau+1}{2\tau^2}ln\frac{\tau/\tau_m}{1+\tau}), \quad (12)$$

Intrabeam scattering

Piwinski and Bjorken-Mtingwa IBS growth rates

PETRA IV. NEW DIMENSIONS

Piwinksi (1974) classical Rutherford scattering – weak focussing (extended in the CIMP)
Bjorken-Mtingwa (1983) quantum relativistic cross section: expressions look rather different!

$$\frac{1}{T_p} \equiv \frac{1}{\sigma_p}\frac{d\sigma_p}{dt}$$

$$\frac{1}{T_h} \equiv \frac{1}{\varepsilon_h^{1/2}}\frac{d\varepsilon_h^{1/2}}{dt}$$

$$\frac{1}{T_v} \equiv \frac{1}{\varepsilon_v^{1/2}}\frac{d\varepsilon_v^{1/2}}{dt}$$

$$\frac{1}{T_\delta} \approx 2\pi^{3/2}(\log)A\left\langle \frac{\sigma_H^2}{\sigma_\delta^2}\left(\frac{1}{a}g\left(\tfrac{b}{a}\right)+\frac{1}{b}g\left(\tfrac{a}{b}\right)\right)\right\rangle$$

$$\frac{1}{T_x} \approx 2\pi^{3/2}(\log)A\left\langle \frac{\mathcal{H}_x\sigma_H^2}{\varepsilon_x}\left(\frac{1}{a}g\left(\tfrac{b}{a}\right)+\frac{1}{b}g\left(\tfrac{a}{b}\right)\right)-a\,g\left(\tfrac{b}{a}\right)\right\rangle$$

$$\frac{1}{T_y} \approx 2\pi^{3/2}(\log)A\left\langle \frac{\mathcal{H}_y\sigma_H^2}{\varepsilon_y}\left(\frac{1}{a}g\left(\tfrac{b}{a}\right)+\frac{1}{b}g\left(\tfrac{a}{b}\right)\right)-b\,g\left(\tfrac{a}{b}\right)\right\rangle$$

$$\mathcal{H}_x = \gamma_x\eta_x^2 + 2\alpha_x\eta_x\eta_{px} + \beta_x\eta_{px}^2$$

$$\frac{1}{\sigma_H^2} = \frac{1}{\sigma_\delta^2} + \frac{\mathcal{H}_x}{\varepsilon_x} + \frac{\mathcal{H}_y}{\varepsilon_y}$$

$$a = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_x}{\varepsilon_x}} \qquad b = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_y}{\varepsilon_y}}$$

$$g(\omega) = \sqrt{\frac{\pi}{\omega}}\left[P_{-1/2}^{0}\left(\frac{\omega^2+1}{2\omega}\right) \pm \frac{3}{2}P_{-1/2}^{-1}\left(\frac{\omega^2+1}{2\omega}\right)\right]$$

$$(\log) \approx \ln\left(\frac{\gamma^2\sigma_y\varepsilon_x}{r_e\beta_x}\right)$$

CIMP

$$\frac{1}{T_i} = 4\pi A(\log)\left\langle \int_0^\infty d\lambda \frac{\lambda^{1/2}}{[\det(L+\lambda I)]^{1/2}} \times \left\{\mathrm{Tr}L^i\,\mathrm{Tr}\left(\frac{1}{L+\lambda I}\right) - 3\,\mathrm{Tr}\left[L^i\left(\frac{1}{L+\lambda I}\right)\right]\right\}\right\rangle$$

$$L = L^{(p)} + L^{(h)} + L^{(v)} \qquad L^{(p)} = \frac{\gamma^2}{\sigma_p^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L^{(h)} = \frac{\beta_h}{\varepsilon_h}\begin{pmatrix} 1 & -\gamma\phi_h & 0 \\ -\gamma\phi_h & \frac{\gamma^2\mathcal{H}_h}{\beta_h} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L^{(v)} = \frac{\beta_v}{\varepsilon_v}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\gamma^2\mathcal{H}_v}{\beta_v} & -\gamma\phi_v \\ 0 & -\gamma\phi_v & 1 \end{pmatrix}$$

$$\mathcal{H}_h = [\eta_h^2 + (\beta_h\eta_h' - \tfrac{1}{2}\beta_h'\eta_h)^2]/\beta_h$$
$$\phi_h = \eta_h' - \tfrac{1}{2}\beta_h'\eta_h/\beta_h$$

BM

$$A = \frac{r_0^2 cN}{64\pi^2\beta^3\gamma^4\varepsilon_h\varepsilon_v\sigma_s\sigma_p}$$

(log) is the Coulomb log factor
log (b_{max}/b_{min})
typically ~8-16

Growth rates are obtained from integrals of local growth rates along the ring

Intrabeam scattering

Equilibrium emittance with IBS

The evolution of the emittance has an additional contribution due to IBS

The equations must be solved in a self consistent way as $T_{x,y,p} = T_{x,y,p}(\varepsilon_x, \varepsilon_y, \sigma_p)$

$$\frac{d\varepsilon_x}{dt} \equiv -\frac{2}{\tau_x}(\varepsilon_x - \varepsilon_{x0}) + \frac{2\varepsilon_x}{T_x} = 0$$

$$\frac{d\varepsilon_y}{dt} \equiv -\frac{2}{\tau_y}(\varepsilon_y - \varepsilon_{x0}) + \frac{2\varepsilon_y}{T_y} = 0$$

$$\frac{d(\sigma_p^2)}{dt} \equiv -\frac{2}{\tau_p}(\sigma_p^2 - \sigma_{p0}^2) + \frac{2\sigma_p^2}{T_p} = 0$$

Equilibrium solutions found for d/dt = 0

$$\varepsilon_x = \frac{\varepsilon_{x,0}}{1-\tau_x/T_x(\varepsilon_x,\varepsilon_y,\sigma_p)}$$

$$\varepsilon_y = \frac{\varepsilon_{y,0}}{1-\tau_y/T_y(\varepsilon_x,\varepsilon_y,\sigma_p)}$$

$$\sigma_p^2 = \frac{\sigma_{p,0}^2}{1-\tau_p/T_p(\varepsilon_x,\varepsilon_y,\sigma_p)}$$

The IBS equilibrium emittance depends on

- charge density
- beam energy
- lattice optics and beam parameters (emittance, energy spread, bunch length)
- ratio τ_x/T_x the relative magnitude of IBS growth rates and radiation damping rates

Intrabeam scattering

Intrabeam scattering analysis of measurements at KEK's Accelerator Test Facility damping ring

Bane IBS growth rates

Bane (2002) has proven that the Piwinski and BM IBS growth rates can be cast in the same way in the limit of high energy beams

Longitudinal IBS growth rate: $$\frac{1}{T_p} \approx \frac{r_e^2 c N_b (\log)}{16\gamma^3 \epsilon_x^{3/4} \epsilon_y^{3/4} \sigma_z \sigma_p^3} \left\langle \sigma_H \, g(a/b) \, (\beta_x \beta_y)^{-1/4} \right\rangle$$

Transverse IBS growth rate: $$\frac{1}{T_{x,y}} \approx \frac{\sigma_p^2 \langle \mathcal{H}_{x,y} \rangle}{\epsilon_{x,y}} \frac{1}{T_p} ,$$

$$\frac{1}{\sigma_H^2} = \frac{1}{\sigma_p^2} + \frac{\mathcal{H}_x}{\epsilon_x} + \frac{\mathcal{H}_y}{\epsilon_y} \qquad a = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_x}{\epsilon_x}} \qquad b = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_y}{\epsilon_y}} \qquad g(\alpha) = \alpha^{(0.021 - 0.044 \ln \alpha)}$$

In all cases the growth rates depend on the local optics functions and have to be averaged around the ring

$$H_x = \gamma \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

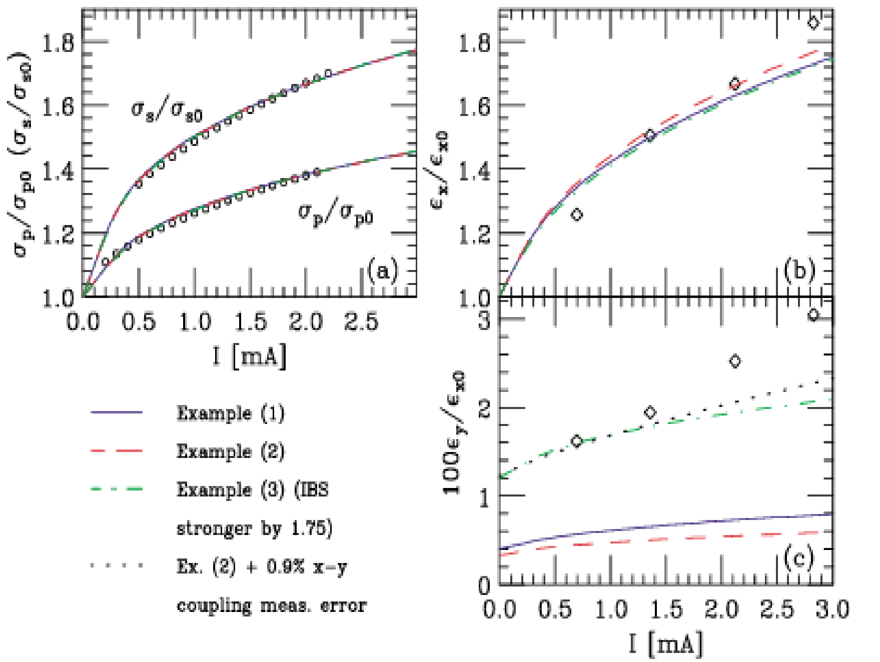


FIG. 6. (Color) ATF measurement data (symbols) and IBS theory fits (the curves). The symbols in (a) give the smooth curve fits to the measured data of Fig. 5.

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Intrabeam scattering

Experimental tests of IBS emittance growth: SLS

The SLS is an ideal test facility for IBS studies:

- Record vertical emittance of 1 pm-rad at nominal energy (2.4 GeV)
- Availability of emittance monitoring diagnostics (hor./vert. beam size monitors)
- Ability to run at lower energies

The IBS effect at nominal energy is weak even at high bunch currents while at low energy is strongly enhanced and can be measured even at low bunch currents

Good agreement with CIMP theory found in transverse plane (longitudinal not measured)