Homework #3

2025 Accelerator Summer School

Due Aug. 8 (Fri.), 9:30 AM, 2025

1. (a) Show that the eigenvalues $\lambda_\pm$ of the matrix $\Sigma \cdot S$ where

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1}$$

are $\lambda_\pm = \pm i\epsilon_x$, where ϵ_x is the beam emittance.

(b) Consider a 2×2 transfer matrix R given by

$$R = I \cos\mu_x + S \cdot A_x \sin\mu_x \tag{2}$$

where I is the 2×2 identity matrix, μ_x is a real number, and the matrix A_x is constructed from the Twiss parameters

$$A_x = \begin{pmatrix} \gamma_x & \alpha_x \\ \alpha_x & \beta_x \end{pmatrix} \tag{3}$$

Show that, if the second-order moments of the beam distribution satisfy equations

$$\begin{aligned} \langle x^2 \rangle &= \beta_x \epsilon_x, \\ \langle x p_x \rangle &= -\alpha_x \epsilon_x, \\ \langle p_x^2 \rangle &= \gamma_x \epsilon_x, \end{aligned}$$

then the beam distribution (i.e. beam envelope matrix or beam sigma matrix) is "matched" to the transfer matrix, i.e. the beam distribution matrix is invariant under the transformation

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \to R \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{4}$$

[Hint: You only need to show that the transformation of the beam sigma matrix satisfies $R \cdot \Sigma \cdot R^T = \Sigma$.]

2. (a) As in Eq. (34) of Lecture 1 slide, Linear Dynamics], the (scaled) energy deviation p_t for a particle of rest mass m is defined by (in MAD notation)

$$p_t = \frac{E}{cp_0} - \frac{1}{\beta_0} = \frac{\gamma mc}{p_0} - \frac{1}{\beta_0} \tag{5}$$

where γ is the relativistic factor for the particle, p_0 is the reference momentum, and $\beta_0 c$ is the velocity of a particle. Show that

$$\beta\gamma = \beta_0\gamma_0 \sqrt{1 + \frac{2p_t}{\beta_0} + p_t^2} \tag{6}$$

where βc is the velocity of a particle with energy deviation p_t, and γ_0 is the relativistic factor for a particle with rest mass m and momentum equal to the reference momentum.

(b) Consider a particle moving in a horizontal plane in a uniform vertical magnetic field. The region of the field is large enough for the trajectory of the particle to describe a complete circle. Using the result from Problem 2(a), show that the circumference C of the trajectory as a function of the (scaled) energy deviation p_t is given by

$$\frac{\Delta C}{C_0} = \sqrt{1 + \frac{2p_t}{\beta_0} + p_t^2} - 1 \tag{7}$$

where

$$\Delta C = C - C_0 \tag{8}$$

and C_0 is the circumference of the trajectory when $p_t = 0$.

3. Show that the electric field and magnetic field of an ultra-relativistic charged particle with charge q is given by (in CGS unit)

$$E_r(s,t) = \frac{2q}{r}\delta(s - ct), \tag{9}$$

$$B_\phi(s,t) = \frac{2q}{r}\delta(s - ct). \tag{10}$$

Hint) Use Maxwell eqs. $\nabla \cdot E = 4\pi\rho, \nabla \times B = \frac{1}{c}\left(4\pi j + \frac{\partial E}{\partial t}\right)$.

4. The longitudinal impedance $Z^{\|}$ of an RLC resonator circuit is given by

$$Z^{\|} = \frac{R_s}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)}, \tag{11}$$

where

$$Q = R_s\sqrt{\frac{C}{L}},\ \omega_R = \sqrt{\frac{1}{LC}}. \tag{12}$$

Show that the wake function is

$$W'(z) = 2\alpha R_s e^{\alpha z/c}\left(\cos\frac{\bar{\omega}z}{c} + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}z}{c}\right),\ z < 0, \tag{13}$$

where

$$\alpha = \frac{\omega_R}{2Q},\ \bar{\omega} = \sqrt{\omega_R^2 - \alpha^2}. \tag{14}$$

Hint) Use solutions $\omega^2 - \omega_R^2 - \omega\omega_R/iQ = 0$ in integral of below equation :
$W'(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega z/c} Z_{\|}(\omega) \propto \int_{-\infty}^{\infty} d\omega e^{i\omega z/c}\left(\frac{\omega_2}{\omega - \omega_2} - \frac{\omega_1}{\omega - \omega_1}\right)$.

5. Show that longitudinal and transverse impedances have the following properties:

$$Z_{\|}^*(\omega) = Z_{\|}(\omega) \tag{15}$$

$$Z_{\perp}^*(\omega) = -Z_{\perp}(-\omega) \tag{16}$$