Potential-well distortion

Microwave instability

Reference

- Andy Wolski, Lect.8 Classical single-bunch instabilities USPAS 2007
- A. Chao, Physics of collective beam instabilities in high energy accelerators, 1993.

Equilibrium charge distribution in an electron storage ring

- We shall discuss how wake fields in a storage ring can change the longitudinal distribution (potential well distortion), or lead to a beam instability (microwave instability)
- In an electron storage ring, the combined effects of synchrotron radiation and longitudinal focusing (from the RF cavities) determine the longitudinal distribution of charge within individual bunches.
- Longitudinal wake fields can contribute to the change in energy of particles as a bunch moves around a storage ring. They may be strong enough to distort the equilibrium shape of the bunch (potential well distortion).
- Very strong wake fields can lead to an instability, in which the longitudinal charge distribution fails to reach equilibrium at all.

Equilibrium charge distribution in storage ring

- Averaged over one turn of the storage ring, the rate of change of the longitudinal co-ordinate z (relative to a reference particle at the center of the bunch) is determined by momentum compaction factor α_p and energy deviation $\delta = (E - E_0)/E_0$ of particle:

$$\frac{dz}{ds} = -\alpha_p \delta,$$ (s is longitudinal position along closed orbit)

Momentum compaction factor α_p is determined by the optics :

$$\alpha_p = \frac{1}{C_0} \oint \frac{\eta_x}{\rho} ds$$

C_0 is circumference, η_x is dispersion function, ρ is radius of curvature of the closed orbit

Equilibrium charge distribution in storage ring

- The rate of change of the energy of the particle is given by the energy gain from RF cavities and the energy loss from synchrotron radiation

$$\frac{d\delta}{ds} = \frac{eV_{\mathrm{RF}}}{E_0 C_0} \sin\left(\phi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0 C_0}, \qquad eV_{\mathrm{RF}} \sin\phi_s = U.$$

(V_{RF} is rf voltage, E_0 is beam energy, ϕ_s is synchronous phase, ω is the RF frequency, U is energy lost per turn through synchrotron radiation.)

If z is small, the particle crosses the rf cavities close to the synchronous phase

$$\frac{dz}{ds} = -\alpha_p \delta \quad (1), \qquad \boxed{\frac{d\delta}{ds} = -\frac{eV_{\mathrm{RF}}}{E_0 C_0}\frac{\omega_{\mathrm{RF}}}{c}\cos(\phi_s)\, z \quad (2).}$$

(1) and (2) describe simple harmonic motion in longitudinal phase with angular frequency ω_s

$$\frac{d^2 z}{ds^2} + \frac{\omega_s^2}{c^2} z = 0, \qquad \frac{\omega_s^2}{c^2} = -\frac{eV_{\mathrm{RF}}}{E_0 C_0}\frac{\omega_{\mathrm{RF}}}{c}\alpha_p \cos\phi_s .$$

Equilibrium charge distribution in storage ring

Equations of motion can be obtained from a Hamiltonian

$$H = -\frac{1}{2}\alpha_p \delta^2 - \frac{1}{2\alpha_p}\frac{\omega_s^2}{c^2}z^2, \qquad \frac{dz}{ds} = \frac{\partial H}{\partial \delta}, \qquad \frac{d\delta}{ds} = -\frac{\partial H}{\partial z}.$$

At low bunch charges, where longitudinal wake fields are negligible, the longitudinal charge distribution is usually Gaussian.

$$\Psi(z,\delta) = \Psi_0 \exp\left(\frac{H}{H_0}\right) = \Psi_0 \exp\left(-\frac{z^2}{2\sigma_z^2}\right)\exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right),$$

peak value Ψ_0 at $z = \delta = 0$

H_0 is a constant related to the rms energy spread: $H_0 = \alpha_p \sigma_\delta^2.$

$$\Psi(z,\delta) = \Psi_0 exp\left(\frac{-\frac{1}{2}\alpha_p\delta^2 - \frac{1}{2\alpha_p}\frac{\omega_s^2}{c^2}z^2}{\alpha_p\sigma_\delta^2}\right). \qquad \sigma_z = \frac{\alpha_p c}{\omega_s}\sigma_\delta$$

Effect of wake fields on the equilibrium charge distribution

For a particle of charge e following a particle of charge Ne through the beam line, change in energy of the trailing particle is

$$\Delta\delta = -\frac{Ne^2}{E_0} W_{\parallel}(\Delta z), \qquad (\delta = \frac{\Delta E}{E_0})$$

For a particle within a bunch, we have to sum the contributions from all the "slices" within the bunch ahead of the given particle

$$\Delta\delta(z) = -\frac{e}{E_0}\int_z^{\infty} \lambda(z') W_{\parallel}(z - z') dz' ,$$

($\lambda(z)$ is the longitudinal charge density)

Equations of motion

$$\frac{dz}{ds} = -\alpha_p \delta,$$

$$\frac{d\delta}{ds} = \frac{\omega_s^2}{\alpha_p c^2} z - \frac{e}{E_0 C_0}\int_z^{\infty} \lambda(z') W_{\parallel}(z - z') dz' ,$$

(ω_s : synchrotron frequency in the absence of wakefields.)

Effect of wake fields on the equilibrium charge distribution

$$H_{\mathrm{WF}} = -\frac{1}{2}\alpha_p\delta^2 - \frac{\omega_s^2}{2\alpha_p c^2}z^2 + \frac{e}{E_0 C_0}\int_0^z dz' \int_{z'}^{\infty} dz'' \lambda(z'') W_{\parallel}(z'-z'').$$

Assuming that the particles again have a Gaussian momentum distribution, the charge density in longitudinal phase space

$$\Psi(z,\delta) = \Psi_0 \exp\left(\frac{H}{H_0}\right) = \Psi_0 \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right)\exp\left(-\frac{z^2}{2\sigma_z^2} + \frac{e}{\alpha_p\sigma_\delta^2 E_0 C_0}\int_0^z dz' \int_{z'}^{\infty} dz'' \lambda(z'') W_{\parallel}(z'-z'')\right) \quad (3)$$

$$\lambda(z) = \int_{-\infty}^{\infty} \Psi(z,\delta)\, d\delta,$$

We can integrate both sides of (3) with respect to δ, to obtain an integral equation for $\lambda(z)$, known as the Haissinski equation:

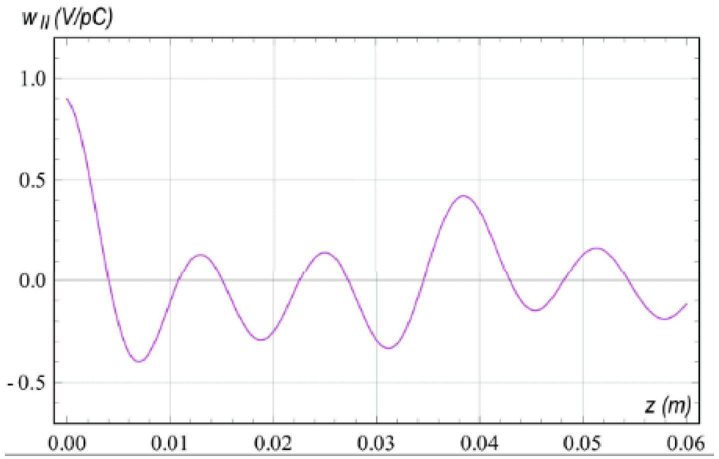
$$\lambda(z) = \lambda_0 \exp\left(-\frac{z^2}{2\sigma_z^2} + \frac{e}{\alpha_p\sigma_\delta^2 E_0 C_0}\int_0^z dz' \int_{z'}^{\infty} dz'' \lambda(z'') W_{\parallel}(z'-z'')\right),$$

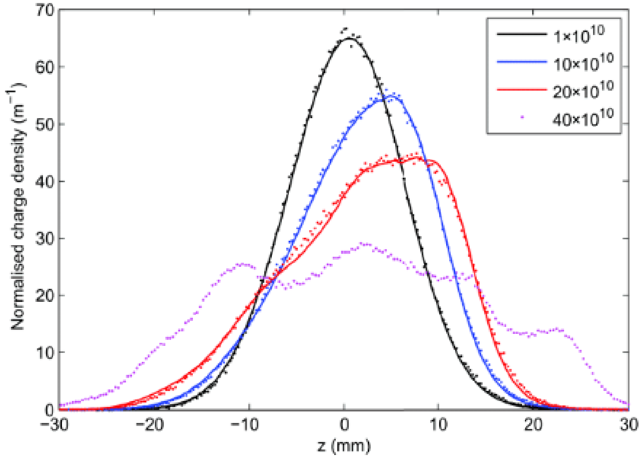
Constant λ_0 is determined by the condition that the integral over $\lambda(z)$ is equal to the total charge in the bunch,

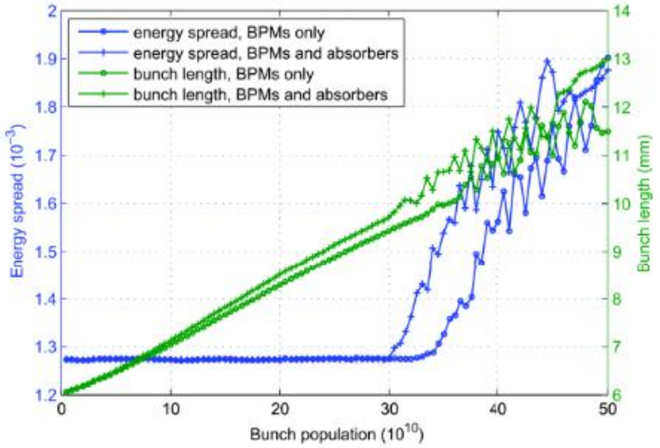
$$\int_{-\infty}^{\infty} \lambda(z)\, dz = Ne,$$

Example: potential well distortion in the ILC damping rings (due to BPM)

- With the wake function, the equilibrium bunch profile can be found for different bunch charges by solving (numerically) the Haissinski equation





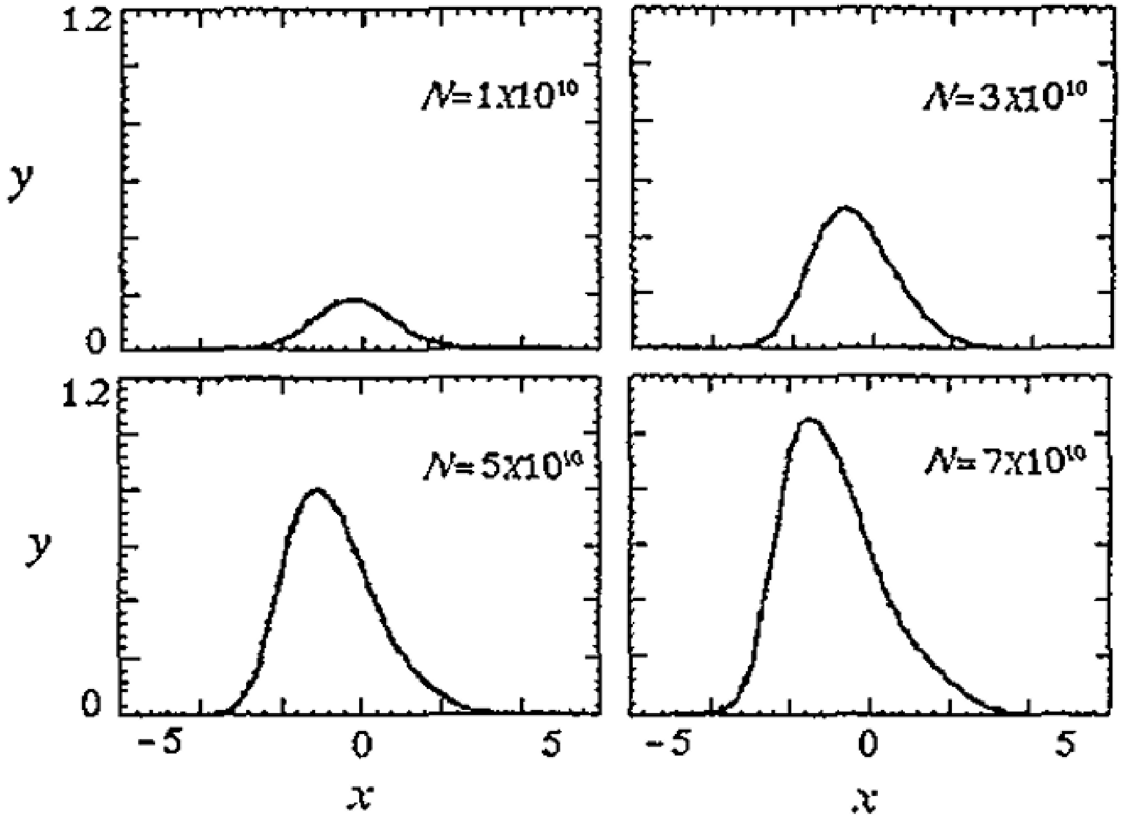


The solutions to the Haissinski equation (solid lines in center plot) can be compared with the results from particle tracking (dots). At high bunch charges, no equilibrium solution exists: the bunch is unstable.

The rms energy spread remains constant up to a bunch population of 3×10^{11} particles: if the population is increased beyond this point, the bunch becomes unstable (fails to reach an equilibrium).

Example: Potential Well Distortion in the KEK-ATF Damping Ring

$$y = IZ_o/(\dot{V}\sigma_o)/10^8$$



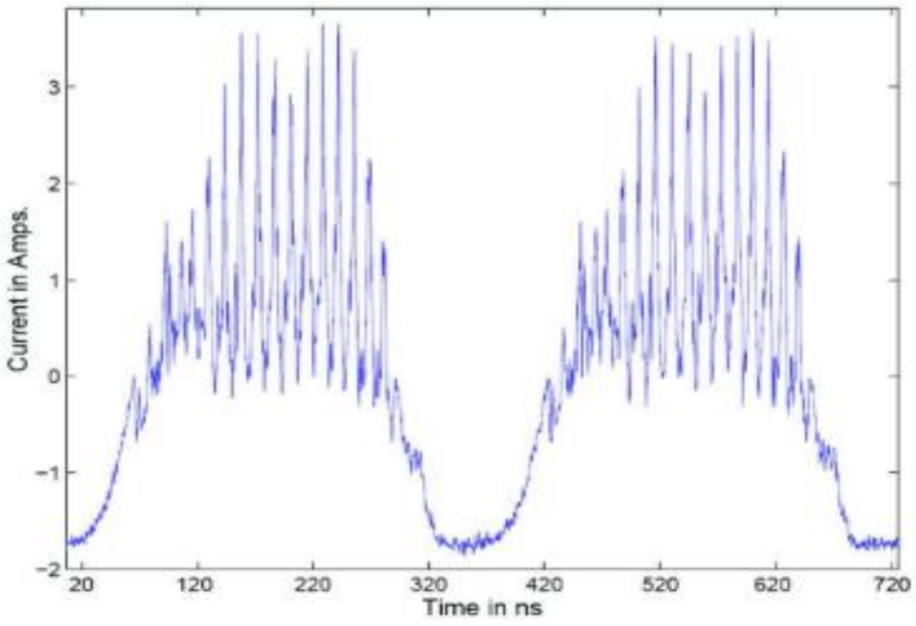
$$x = t/\sigma_o.$$

Calculated bunch distribution due to potential well distortion for various beam intensities in KEK-ATF damping ring

KEK Preprint 98-21 Eun-San Kim

Microwave instability

- We shall discuss how wakefields can lead to instability in the longitudinal phase space distribution of charge in individual bunches in a storage ring.
- "Microwave instability" is characterized by the appearance of structures within a bunch on a scale small compared to the overall bunch length.



Observation of single-bunch longitudinal instability in the Los Alamos PSR, caused by an inductive impedance.

From C. Beltran, A.A. Browman and R.J. Macek, "Calculations and observations of the longitudinal instability caused by the ferrite inductors at the Los Alamos Proton Storage Ring", Proceedings of the 2003 Particle Accelerator Conference, Portland, Oregon.

Liouville's theorem and the Vlasov equation

• To understand the behaviour of a bunch of particles that is not in equilibrium, we need an equation describing the dynamics of the charge distribution within the bunch.

• An appropriate description is provided by the Vlasov equation, which may be "derived" from Liouville's theorem.

• Let us consider the longitudinal phase space, with co-ordinate $\theta = 2\pi s / C_0$ (where C_0 is the circumference) and conjugate momentum δ (energy deviation of a particle)

Liouville's theorem $$\frac{\partial \Psi}{\partial s} + \frac{d\theta}{ds}\frac{\partial \Psi}{\partial \theta} + \frac{d\delta}{ds}\frac{\partial \Psi}{\partial \delta} = 0$$

Single-bunch beam instabilities can be described by non-stationary solutions to the Vlasov equation.

Perturbation approach to the Vlasov equation

Steps appropriate for an analysis of the microwave instability are as follows:

1. Assume an initial phase space distribution of the form:

$$\Psi(\theta, \delta; t) = \Psi_0(\delta) + \Delta\Psi\, e^{i(n\theta - \omega_n t)}$$

: $\Psi_0(\delta)$ is a stationary (equilibrium) distribution, and $\Delta\Psi$ is amplitude of a density modulation with "wavelength " C_0/n and oscillation frequency ω_n.

2. Substitute the distribution into the Vlasov equation, and expand each term to first order in $\Delta\Psi$
3. Solve the resulting equation for frequency of oscillation of perturbation, ω_n.

If there is a solution for ω_n with a positive imaginary part, then amplitude of the perturbation will grow exponentially : this indicates an instability.

The dispersion relation

- The rate at which a particle moves around the ring depends on the momentum compaction factor α_p and energy deviation of the particle δ.

$$\frac{d\theta}{ds} = \omega = \omega_0(1 - \alpha_p\delta) \qquad (C = C_0\ (1 + \alpha_p\delta))$$

- The change in energy of a particle resulting from the longitudinal wake fields is given by the convolution of the current spectrum with the longitudinal impedance.

- Let us assume that the charge distribution around the ring is described by a sinusoidal modulation, superposed on a uniform distribution.

  Beam current observed at a point $\theta = 2\pi s/C_0$ in the ring is given by

$$I(\theta, t) = I_0 + \Delta I\ e^{i(n\theta - \omega_n t)}.$$

- Change in energy of a particle in one revolution of the ring

$$\Delta E = -e\,\Delta I\ Z_{\parallel}(\omega_n)\ e^{i(n\theta - \omega_n t)}$$

The dispersion relation

If the beam distribution in phase space is normalized, $\int_{-\infty}^{\infty} \Psi_0 d\delta = 1$,

Then the amplitude of the current modulation is $\Delta I = I_0 \int_{-\infty}^{\infty} \Delta\Psi \, d\delta$.

The rate of change of the energy deviation is

$$\Delta E = -Z_{\parallel}(\omega_n) e I_0 \int_{-\infty}^{\infty} \Delta\Psi \, d\delta \, e^{i(n\theta - \omega_n t)}$$

$$\frac{d\delta}{ds} = \frac{\Delta E}{C_0 E_0} = -Z_{\parallel}(\omega_n) \frac{e I_0}{C_0 E_0} \int_{-\infty}^{\infty} \Delta\Psi \, d\delta \, e^{i(n\theta - \omega_n t)}$$

$$\frac{\partial\Psi}{\partial s} + \frac{d\theta}{ds}\frac{\partial\Psi}{\partial\theta} + \frac{d\delta}{ds}\frac{\partial\Psi}{\partial\delta} = 0, \qquad \Psi(\theta, \delta; t) = \Psi_0(\delta) + \Delta\Psi e^{i(n\theta - \omega_n t)}. \qquad \left(\frac{\partial\Psi}{\partial t} + \frac{d\theta}{dt}\frac{\partial\Psi}{\partial\theta} + \frac{d\delta}{dt}\frac{\partial\Psi}{\partial\delta} = 0\right)$$

$$\Delta\Psi = -iZ_{\parallel}(\omega_n) \frac{c e I_0}{C_0 E_0} \int_{-\infty}^{\infty} \Delta\Psi d\delta \frac{\partial\Psi_0/\partial\delta}{(n\omega - \omega_n)} \qquad \left(\frac{d\theta}{dt} = \omega\right)$$

We integrate both sides of this equation over δ.

$$1 = -iZ_{\parallel}(\omega_n) \frac{c e I_0}{C_0 E_0} \int_{-\infty}^{\infty} \frac{\partial\Psi_0/\partial\delta}{(n\omega - \omega_n)} d\delta \qquad \omega = \omega_0(1 - \alpha_p \delta)$$

Dispersion relation

$$1 = -iZ_{\parallel}(\omega_n)\frac{ceI_0}{C_0E_0}\int_{-\infty}^{\infty}\frac{\partial\Psi_0/\partial\delta}{(n\omega-\omega_n)}d\delta$$

I_0 is the beam current, C_0 is the circumference, E_0 is the beam energy, $\omega = \omega_0\left(1-\alpha_p\delta\right)$ is revolution frequency, α_p is momentum compaction factor and $Z_{\parallel}(\omega_n)$ is longitudinal impedance.

Equation relates the wavelength of the density modulation (characterized by the "mode number" n) to the frequency of the modulation: it is known as the dispersion relation.

In practice, as a result of random fluctuations in the particle density, all modes will be present to some extent. If there exists a mode n for which the frequency ω_n has a positive imaginary part, then the beam is likely to be unstable.

Because we have retained terms in the Vlasov equation only up to first order in the perturbation $\Delta\Psi$, the dispersion relation can only give an indication of whether the beam is stable or not: it cannot be used to describe the behaviour of the beam if an instability is present.

Example 1: "cold" beam

- Consider the case of a "cold" beam, i.e. a beam with zero energy spread.
- Energy spread is described by a Dirac delta function: the energy distribution function $\Psi_0(\delta)$ is zero, except for $\delta = 0$. Integrating by parts, and using $\omega = \omega_0\left(1 - \alpha_p \delta\right)$

$$\int_{-\infty}^{\infty} \frac{\partial \Psi_0/\partial\delta}{(n\omega - \omega_n)} d\delta = \int_{-\infty}^{\infty} \frac{\Psi_0}{(n\omega - \omega_n)^2} n \frac{\partial\omega}{\partial\delta} d\delta = -\frac{n\omega_0\alpha_p}{(n\omega - \omega_n)^2}, \qquad (\Psi_0(\delta) = \delta(\delta))$$

$$(n\omega_0 - \omega_n)^2 = iZ_{\parallel}(\omega_n)\frac{I_0}{E_0/e}\frac{n\omega_0^2\alpha_p}{2\pi},$$

$$\frac{\omega_n}{n\omega_0} = 1 \pm \sqrt{i\frac{Z_{\parallel}(\omega_n)}{n}\frac{I_0}{E_0/e}\frac{\alpha_p}{2\pi}}$$

There is always a solution for ω_n with positive imaginary part unless impedance is a purely imaginary number.

Hence, a beam with zero energy spread will always be unstable in the presence of any longitudinal impedance.

Landau damping

- In practice, there are some spread in energy for the particles in a storage ring.
- Combined with the (non-zero) momentum compaction of the lattice, the energy spread will lead to a range in revolution frequency for the particles in the beam.
- The spread in revolution frequencies means that any density modulation will tend to get "smeared out", leading to a reduction in the amplitude of the density modulation.
- If the rate of reduction in amplitude of the density modulation is sufficient to suppress the growth in amplitude from the impedance, then the beam will be stable.
- The suppression of the beam instability arising from the spread in energy of particles in the beam is known as "Landau damping"

Example 2: a beam with a Gaussian energy spread

- As an example of the effect of Landau damping, let us consider the case of a beam with Gaussian energy spread

$$\Psi_0 = \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi}\sigma_\delta} \qquad 1 = -iZ_\parallel(\omega_n)\frac{ceI_0}{C_0E_0}\int_{-\infty}^{\infty}\frac{\partial\Psi_0/\partial\delta}{(n\omega - \omega_n)}d\delta$$

from δ to $\zeta = \delta/\sigma_\delta$

$$1 = i\frac{Z_\parallel(\omega_n)}{n}\frac{I_0}{(2\pi)^{3/2}(E_0/e)\alpha_p\sigma_\delta^2}\int_{-\infty}^{\infty}\frac{\zeta e^{-\zeta^2/2}}{\zeta + \Delta_n}d\zeta, \qquad \Delta_n = \frac{\omega_n - n\omega_0}{n\omega_0\alpha_p\sigma_\delta}$$

We are only really interested in whether the beam is stable or not, and this can be determined from the imaginary part of ω_n; and hence, from the imaginary part of Δ_n

Example 2: a beam with a Gaussian energy spread

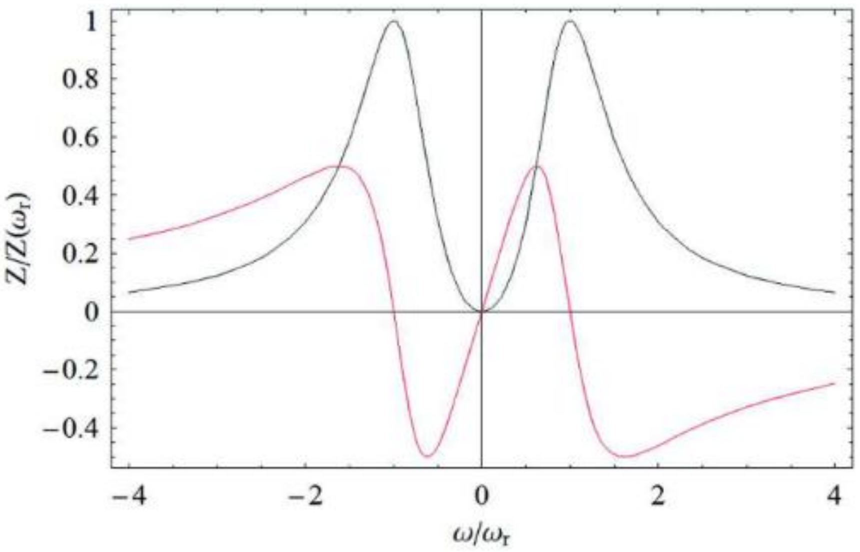
To apply the dispersion relation to determine the stability of the beam, we write the dispersion relation in the form

$$F(n) = U + iV, \qquad 1 = iF(n)\int_{-\infty}^{\infty} \frac{\zeta e^{-\zeta^2/2}}{\zeta + \Delta_n} d\zeta$$

$$F(n) = \frac{Z_\parallel(\omega_n)}{n} \frac{I_0}{(2\pi)^{3/2}(E_0/e)\alpha_p \sigma_\delta^2} \qquad U + iV = \left(i\int_{-\infty}^{\infty} \frac{\zeta e^{-\zeta^2/2}}{\zeta + \Delta_n} d\zeta \right)^{-1}$$

- As an example, consider the case of a storage ring with a broad-band impedance, with characteristic frequency ω_r

$$Z_\parallel(\omega) = Z_\parallel(\omega_r) \frac{1 - i\dfrac{\omega^2 - \omega_r^2}{\omega_r \omega}}{1 + \dfrac{(\omega^2 - \omega_r^2)^2}{\omega_r^2 \omega^2}}$$

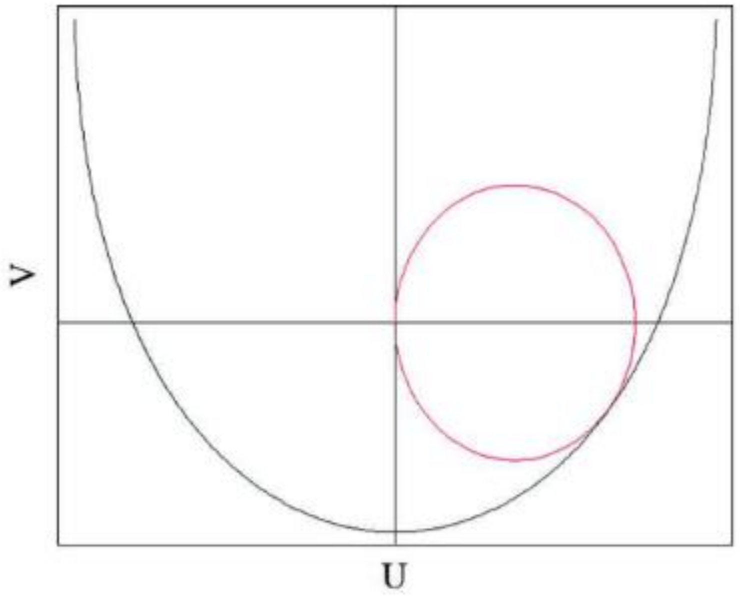


Example 2: a beam with a Gaussian energy spread

$$U + iV = \left(i \int_{-\infty}^{\infty} \frac{\zeta e^{-\zeta^2/2}}{\zeta + \Delta_n} d\zeta \right)^{-1}$$

If we consider the case that Im(Δ_n) >> 0 (i.e. Δ_n has a very large positive part), U + iV will be a large real number, i.e. V will be close to zero and U will be very large. It is outside area enclosed by the black curve in the plot.

The case U = V =0 (i.e. at the center of the plot, inside the black curve) can only occur in the limit of low beam current or zero impedance: in that case, the beam has to be stable. So the stable region is inside the black curve (and has Im(Δ_n) < 0)



plot $F(n)$ for a range of values of n (red curve),and $U + iV$ for a range of real values of Δn (black curve).

If we make the approximation, for a broadband impedance: $Z(\omega_n) \sim Z(n\omega_o)$, then the red curve is close to U + iV for different values of n. So the red and black curves touch when U ~ $\pi/6$

$$F(n) = \frac{Z_{\parallel}(\omega_n)}{n} \frac{I_0}{(2\pi)^{3/2}(E_0/e)\alpha_p \sigma_\delta^2} \sim \frac{\pi}{6}$$

Example 2: a beam with a Gaussian energy spread

The instability threshold corresponds to the condition

$$I_0 = \frac{\pi^2\sqrt{2\pi}}{3}\alpha_p\sigma_\delta^2\frac{E_0/e}{Z_\parallel(\omega_r)/n}. \quad (\omega_n = n\omega_0 = \omega_r)$$

This represents the maximum current that can be injected into the storage ring while maintaining beam stability. We see that we can raise the instability threshold by:

- increasing the momentum compaction factor or the energy spread: this increases the rate of Landau damping;
- increasing the beam energy : this increases the beam rigidity;
- reducing the impedance

Application to bunched beams

In applying the stability criterion to a bunched beam, we should replace the **average current I_0** by the peak current $\hat{I}$, which for a Gaussian bunch is:

$$\hat{I} = \frac{ecN_0}{\sqrt{2\pi}\sigma_z},$$

For bunched beams, the stability criterion can be written (as a limit on the impedance)

$$\frac{Z_\parallel(\omega_r)}{n} < \frac{\pi^2}{6} Z_0 \frac{\gamma \alpha_p \sigma_\delta^2 \sigma_z}{r_e N_0},$$

A further commonly used approximation is to replace the stability boundary obtained from Im(Δn) = 0 (black curve) with a circle of radius 1/√2π (red curve)

$$F(n) = \frac{Z_\parallel(\omega_n)}{n} \frac{I_0}{(2\pi)^{3/2}(E_0/e)\alpha_p \sigma_\delta^2} < 1/\sqrt{2\pi}$$

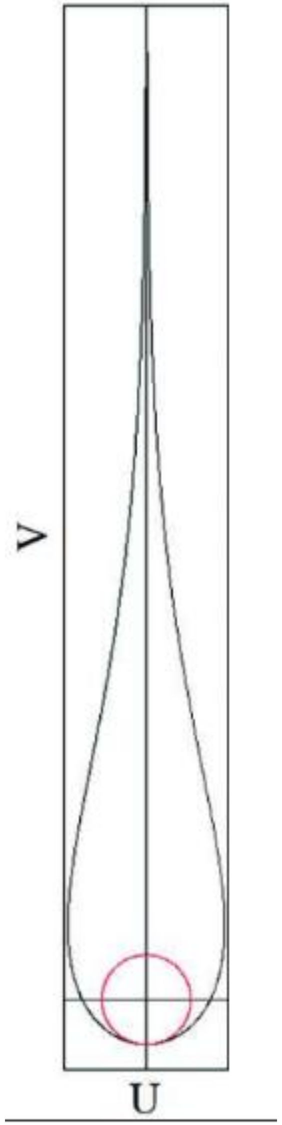
The stability criterion for coasting beams

$$\left|\frac{Z_\parallel(\omega_r)}{n}\right| < 2\pi \frac{E_0/e}{I_0} \alpha_0 \sigma_\delta^2,$$ Keil–Schnell criterion

For bunched beams, the stability criterion is

$$\left|\frac{Z_\parallel(\omega_r)}{n}\right| < \sqrt{\frac{\pi}{2}} Z_0 \frac{\gamma \alpha_p \sigma_\delta^2 \sigma_z}{r_e N_0},$$ Keil–Schnell–Boussard criterion

average current I_0 by the peak current



Characteristics of microwave instability

- The type of instability that we have described often appears as a modulation in charge density within individual bunches in a storage ring, on a length scale of order of a millimeter.

- The charge density modulation can lead to the emission of detectable microwave radiation: the instability is therefore often known as the "microwave instability".

- Since our analysis is based on linearizing the Vlasov equation (i.e. keeping terms only up to first order in the density modulation $\Delta\Psi$), we can only estimate the threshold of the instability: we cannot describe how the beam behaves above threshold.

Characteristics of microwave instability

- Further theoretical analysis, together with numerical modelling and experimental studies, indicate that above the instability threshold there is an increase in beam energy spread following a 1/3 power law:

$$\sigma_\delta = \sigma_{\delta,0} + k(N_0 - N_{\mathrm{th}})^{\frac{1}{3}},$$

$\sigma_{\delta,0}$ is the natural energy spread, N_0 is the bunch population and N_{th} is bunch population at the instability threshold.

- The increase in energy spread leads to an increase in the bunch length.
- The microwave instability is also known as "turbulent bunch lengthening"

Summary : Potential well distortion

- In an electron storage ring in the absence of wake fields, beam generally has a Gaussian distribution in longitudinal phase space.

- Wake fields can drive beam instabilities; but at low currents (below instability threshold) beam distribution can still reach an equilibrium.

- Below instability threshold, longitudinal wake fields have little impact on the energy spread, but the longitudinal charge profile within a bunch can be changed: this effect is known as potential well distortion.

- The equilibrium charge profile in the presence of longitudinal wake fields is described by the Haissinski equation.

Summary : Microwave instability

- At high bunch currents, short-range wake fields drive beam instabilities where the charge within the bunch fails to reach an equilibrium.

- The dynamics of the charge distribution in longitudinal phase space for a single bunch is described by the Vlasov equation.

- With some approximations and assumptions, it is possible to find a solution to the Vlasov equation that relates the frequency of a small modulation on the charge density to the wavelength of the modulation: the equation describing this relationship is known as the dispersion relation.
- The stability of a modulation of given wavelength is determined by the imaginary part of the oscillation frequency of the modulation, which can be found from the dispersion relation.

Summary : Microwave instability

- A "cold" beam (with zero energy spread) is always unstable in the presence of longitudinal wake fields.
- When the energy spread is non-zero, the effects of momentum compaction leads to particles moving round the ring at different rates, depending on their energy deviation.
- As a result, with non-zero energy spread (and non-zero momentum compaction) any modulation in charge density tends to get "smeared out", suppressing the development of a beam instability: this process is known as **Landau damping**.