Coupled-Bunch Instabilities

Reference

- Andy Wolski, Classical coupled-bunch instabilities, USPAS 2007
- A. Chao, Physics of collective beam instabilities in high energy accelerator, 1993

Wake fields and wake functions

- Fields generated by the head of a bunch can act back on particles at the tail, modifying their dynamics and driving instabilities.
- The electromagnetic fields generated by a particle or a bunch of particles moving through a vacuum chamber are usually described as *wake fields*.
- The wake function gives the effect of a leading particle on a following particle, as a function of the longitudinal distance between the two particles.

Longitudinal and transverse wake functions

- Change in energy of particle B from the wake field of particle A, when the particles move through a given accelerator component, is:

$$\Delta\delta_B = -\frac{r_e}{\gamma} N_A W_{\parallel}(z - z')$$

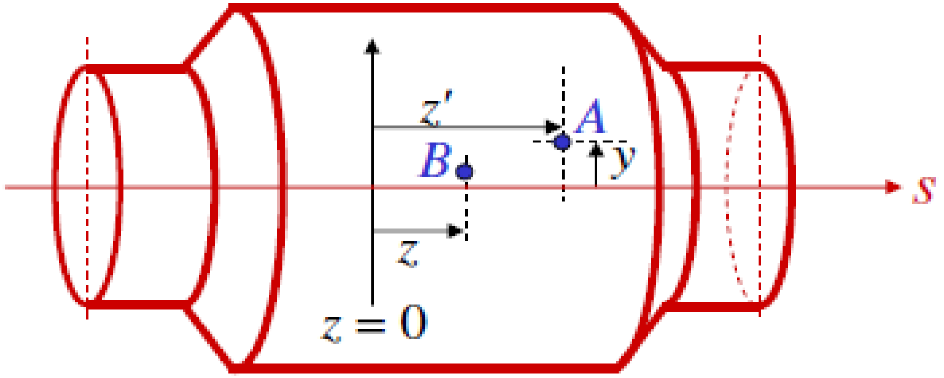
$W_{\parallel}$ is the wake function of the component, eN_A is the charge of particle A
γ is the relativistic factor, r_e is the classical electron radius

- Transverse deflection of a following bunch

$$\Delta p_{y,B} = -\frac{r_e}{\gamma} N_A\, y_A\, W_{\perp}(z - z')$$

y_A: transverse offset of the leading bunch

$W_{\perp}$: transverse wake function



Example: resistive-wall long-range wake functions

- Consider the case of a vacuum chamber with conductivity σ, length *L*, and circular cross-section of radius *b*. Resistive-wall wake fields have both short-range and a long-range effects.

For the parameter regime: $$-z >> \sqrt[3]{\frac{2b^2}{Z_0\sigma}}$$

longitudinal wake function : $$W_{\parallel}(z) = \frac{1}{2\pi b}\sqrt{\frac{4\pi}{Z_0 c}\frac{c}{\sigma}}\frac{L}{\sqrt{-z^3}} \quad \text{(MKS)}$$

transverse wake function : $$W_{\perp}(z) = -\frac{2}{\pi b^3}\sqrt{\frac{4\pi}{Z_0 c}\frac{c}{\sigma}}\frac{L}{\sqrt{-z}} \quad \text{(MKS)}$$

Aluminium has an electrical conductivity (σ) of $3.7\times10^7\ \Omega^{-1}\mathrm{m}^{-1}$;
so for a beam pipe of radius 1 cm, the range of validity of these expressions is $-z \gg 20\ \mu\mathrm{m}$.
they should be safe for studies of multi-bunch effects.

Equation of motion for betatron oscillations

In the absence of any wake fields, the equation of motion for the n^{th} bunch in a storage ring can be written:

$$\ddot{y}_n + {\omega^2}_\beta \, y_n = 0 \qquad\qquad {y_n}'' + \left(\frac{\omega_\beta}{c}\right)^2 y_n = 0$$

betatron frequency as: $\omega_\beta = \dfrac{2\pi \, \nu_\beta}{T_0}$

, ν_β is the betatron tune. We can add the transverse forces from the wake fields as driving terms on the right-hand side of the equation of motion.

Equation of motion with wake fields

If $W_{\perp}(z)$ represents the wake function over the entire circumference, transverse deflection of the n^{th} bunch over one turn can be obtained by summing the wake fields over all bunches over all previous turns:

$$\frac{dp_{y,n}}{dt} = \frac{1}{c}\ddot{y}_n = -\frac{r_e}{T_0\gamma}N_0\sum_k \sum_{m=0}^{M-1} W_\perp\left(-kC - \frac{m-n}{M}C\right)y_m\left(t - kT_0 - \frac{m-n}{M}T_0\right)$$

Ring is uniformly filled with M equally-spaced bunches, each with a total of N_0 particles.

The sum over k represents a sum over multiple turns; the sum over m represents a sum over all the bunches in the ring.

For ultra-relativistic motion, wake function obeys $W_\perp(z) = 0$ if $z > 0$

Equation of motion with wake fields

Including the wake fields, the equation of motion for betatron oscillations can be written:

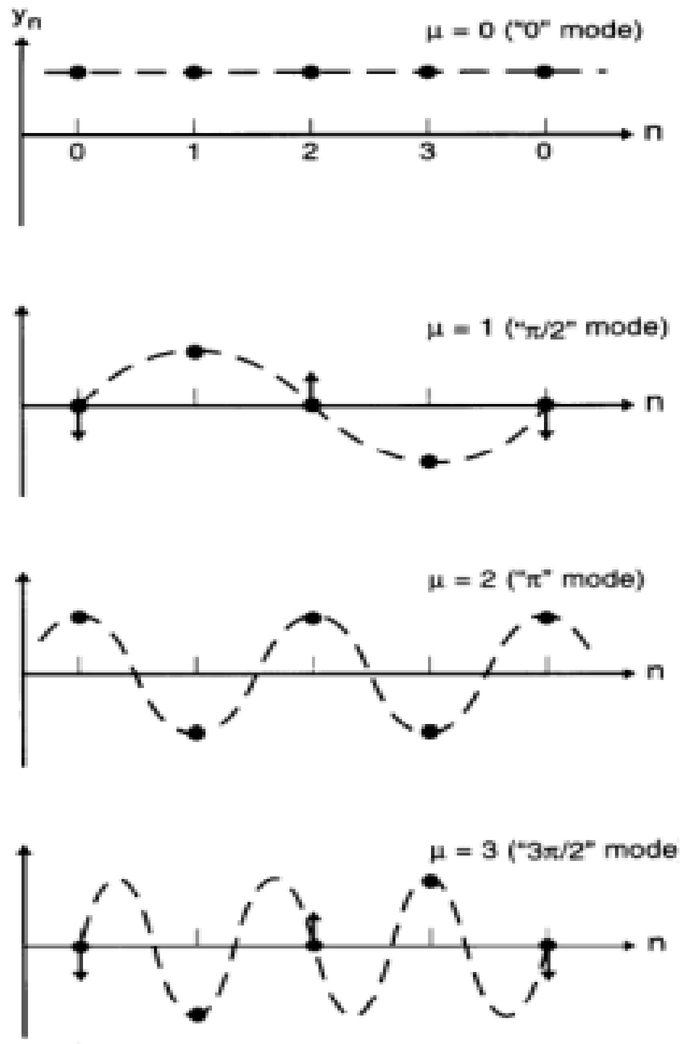
$$\ddot{y}_n + {\omega^2}_\beta\, y_n = -\frac{c r_e}{T_0 \gamma} N_0 \sum_k \sum_{m=0}^{M-1} W_\perp\left(-kC - \frac{m-n}{M} C\right) y_m\left(t - kT_0 - \frac{m-n}{M} T_0\right)$$

We find the behaviour of all the bunches in the ring, in the presence of the long-range wake fields represented by the wake function $W_\perp$. We shall try a solution of the form:

$$y_n(t) \propto exp\left(2\pi i \frac{\mu n}{M}\right) exp(-i\Omega t)$$

This solution describes the behaviour of a "mode" consisting of a particular pattern of transverse bunch positions, and oscillating with a particular frequency. The frequency of a mode μ is represented by Ω_μ ; imaginary part of Ω_μ gives the growth (or damping) rate of the corresponding mode.

Coupled bunch modes



The mode number μ gives the phase advance between the betatron position of one bunch and the next.

Each bunch performs oscillations with frequency Ω_μ as it moves around the ring.

Because the bunches are coupled by the wake fields, the betatron frequency is shifted from the "nominal" frequency ω_β ; the frequency in the presence of the wake fields depends on the mode number.

The real part of Ω_μ - ω_β gives the coherent frequency shift; the imaginary part of Ω_μ - ω_β gives the exponential growth or damping rate for the mode.

Equation of motion with wake fields

$$(-\Omega^2 + \omega_\beta{}^2)\ exp(2\pi i \frac{\mu n}{M}) exp(-i\Omega t)$$

$$= -\frac{N r_0 c}{\gamma T_0} \sum_{m=0}^{M-1} \sum_{k} \exp\left(2\pi i \frac{\mu m}{M}\right)\ W_\perp\ (-kC - \frac{m-n}{M} C) exp(i\Omega T_0 (k + \frac{m-n}{M}))$$

If mode frequency is close to betatron frequency, $\Omega \approx \omega_\beta$. $(\omega_\beta = \omega_o\ \nu_\beta)$

$$(\ \Omega - \omega_\beta) * 2\omega_\beta$$

$$= \frac{N r_0 c}{\gamma T_0} \sum_{m=0}^{M-1} \sum_{k}^{\infty} exp\left(2\pi i \mu \frac{m-n}{M}\right) W_\perp (-kC - \frac{m-n}{M} C) exp(i\Omega T_0 (k + \frac{m-n}{M}))$$

$(\omega_\beta = \omega_o\ \nu_\beta)$

$$(\ \Omega - \omega_\beta)$$

$$= \frac{N r_0 c}{4\pi \nu_\beta\ \gamma} \sum_{m=0}^{M-1} [\sum_{k}^{\infty} W_\perp (-kC - \frac{m-n}{M} C) exp(2\pi i \nu_\beta k)\ ]\ exp(2\pi i (\mu + \nu_\beta) \frac{m-n}{M})$$

Solution of the equation of motion with wake fields

Observe that the factor in square brackets is effectively the Fourier transform of wake function. We define the impedance $Z_{\perp}$ corresponding to the wake field as the Fourier transform of the wake function:

$$Z_{\perp}(\omega) = i\frac{Z_0 c}{4\pi}\int_{-\infty}^{\infty} W_{\perp}(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c}$$

$$W_{\perp}(z) = -i\frac{Z_0 c}{4\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{\perp}(\omega) e^{i\frac{\omega z}{c}} d\omega$$

Solution of the equation of motion with wake fields

From the definition of the impedance

$$\sum_{k=0}^{\infty} W_{\perp}(-kC - \frac{m-n}{M}C)e^{2\pi i\nu_\beta k}$$

$$= -i\frac{4\pi}{Z_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_{\perp}(\omega)\, exp(-i\frac{\omega}{c}(kC + \frac{m-n}{M}C))e^{2\pi i\nu_\beta k}$$

$(\omega_\beta = \omega_o\ \nu_\beta)$

$$= -i\frac{4\pi}{Z_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_{\perp}(\omega)\, exp(-i(\omega - \omega_\beta)T_0 k)\, exp(-i\frac{\omega}{c}(\frac{m-n}{M})C)$$

$$= -i\frac{4\pi}{Z_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_{\perp}(\omega + \omega_\beta) exp(-i\omega T_0 k)\, exp(-i(\omega + \omega_\beta)(\frac{m-n}{M})T_0)$$

Solution of the equation of motion with wake fields

- Note that we can write the summation over k in terms of a Dirac delta function:

$$\sum_{k=0}^{\infty} e^{-i\omega T_0 k} = \sum_{p'=-\infty}^{\infty} \delta\left(\frac{\omega T_0}{2\pi} - p'\right), \quad \left(\sum_{k=-\infty}^{\infty} \delta(x - 2\pi k) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} e^{inx}\right)$$

Then we can perform the integral over ω

$$\sum_{k=0}^{\infty} W_\perp\left(-kC - \frac{m-n}{M}C\right) e^{2\pi i \nu_\beta k}$$

$$= -i \frac{4\pi}{Z_0 c} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_\perp(\omega + \omega_\beta) exp(-i\omega T_0 k)\, exp\left(-i(\omega + \omega_\beta)\left(\frac{m-n}{M}\right) T_0\right)$$

$$= -i \frac{4\pi}{Z_0 c} \frac{1}{T_0} \sum_{p'=-\infty}^{\infty} Z_\perp(p'\omega_0 + \omega_\beta) e^{-i(p'\omega_0 + \omega_\beta)(\frac{m-n}{M} T_0)} \qquad (\omega = p'\omega_0)$$

Solution of the equation of motion with wake fields

$$\Omega_\mu - \omega_\beta$$

$$\approx -i\frac{r_e c}{4\pi\nu_\beta}\frac{N_0}{\gamma}\sum_{m=0}^{M-1}[\frac{4\pi}{Z_0 c}\frac{1}{T_0}\sum_{p\prime=-\infty}^{\infty} Z_\perp(p'\omega_0+\omega_\beta)e^{-i(p\prime\omega_0+\omega_\beta)(\frac{m-n}{M}T_0)}]e^{2\pi i(\mu+\nu_\beta)\frac{m-n}{M}}$$

$(T_0 = 2\pi/\omega_0)$

$$= -i\frac{4\pi}{Z_0 c}\frac{r_e c}{4\pi\nu_\beta}\frac{N_0}{\gamma}\frac{1}{T_0}\sum_{p\prime=-\infty}^{\infty}\sum_{m=0}^{M-1} Z_\perp(p'\omega_0+\omega_\beta)e^{-2\pi i(p\prime-\mu)(\frac{m-n}{M})}$$

We observe that, for large M, the summation over m vanishes, unless:

$p' - \mu = pM$ (p is an integer) $\sum_{m=0}^{M-1} e^{-2\pi i(p\prime-\mu)(\frac{m-n}{M})} = M$

$p' - \mu \neq pM,$ $\sum_{m=0}^{M-1} e^{-2\pi i(\frac{m-n}{M})} = 0$

Solution of the equation of motion with wake fields

$$\Omega_\mu - \omega_\beta \;\approx\; -i\frac{4\pi}{Z_0 c}\frac{MN_0 r_e c}{4\pi\gamma\nu_\beta T_0}\sum_{p\prime=-\infty}^{\infty} Z_\perp[(pM+\mu)\omega_0+\omega_\beta]$$

Ω_μ gives the frequency of a bunch in the case that the bunches are arranged in a mode μ:

$$y_n{}^\mu(t) \;\propto\; exp(2\pi i\frac{\mu n}{M})exp(-i\Omega_\mu t)$$

- We see that associated with long-range wake field, there are two effects:

  - a frequency shift of coherent betatron oscillations, given by the imaginary (“reactive”) part of the impedance

  - an exponential growth or damping of the betatron oscillations, given by the real (“resistive”) part of the impedance.

Solution of the equation of motion with wake fields

Note that we evaluate the impedance at frequencies: $(pM+\mu)\omega_0 + \omega_\beta$

This can be understood in terms of the beam spectrum. At a fixed point in the ring, the beam signal looks like

$$\text{beam signal} \quad \propto \quad \sum_{k=-\infty}^{\infty} \sum_{n=0}^{M-1} y_n{}^{\mu}(t)\, \delta(t - kT_0 + \frac{n}{M} T_0)$$

$$y_n{}^{\mu}(t) \;\propto\; exp\left(2\pi i \frac{\mu n}{M}\right) exp(-i\Omega_\mu t) \approx exp\left(2\pi i \frac{\mu n}{M}\right) exp(-i\omega_\beta t) = \; exp\left(i(2\pi\mu \frac{n}{M} - \omega_\beta t)\right)$$

The beam spectrum is the Fourier transform of the signal

$$spectrum \;\propto \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i\left(2\pi\mu\frac{n}{M} - \omega_\beta t\right)}\, \delta(t - kT_0 + \frac{n}{M} T_0)\, e^{i\omega t} dt$$

Solution of the equation of motion with wake fields

$$spectrum \propto \sum_{k=-\infty}^{\infty} e^{i(\omega-\omega_\beta)kT_0} \sum_{n=0}^{M-1} e^{i\left(2\pi\mu\frac{n}{M}-(\omega-\omega_\beta)\frac{n}{M}T_0\right)}$$

$$\sum_{k=-\infty}^{\infty} e^{i(\omega-\omega_\beta)kT_0} = \omega_0 \sum_{p_0=-\infty}^{\infty} \delta(\omega-\omega_\beta-p_0\omega_0)$$

$$spectrum \propto \omega_0 \sum_{p_0=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i2\pi\mu\frac{n}{M}} e^{-i(\omega-\omega_\beta)\frac{nT_0}{M}} \delta(\omega-\omega_\beta-p_0\omega_0)$$

If $\omega-\omega_\beta \neq p_0\,\omega_0$, the above eq. becomes zero.

$$spectrum = \omega_0 \sum_{p_0=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i2\pi\frac{\mu n}{M}} e^{-ip_0\omega_0\frac{nT_0}{M}} \delta(\omega-\omega_\beta-p_0\omega_0) = \omega_0 \sum_{p_0=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i2\pi\frac{(\mu-p_0)n}{M}} \delta(\omega-\omega_\beta-p_0\omega_0$$

If $\frac{(\mu-p_0)}{M}$ is integer, $\rightarrow$ $e^{i2\pi\frac{(\mu-p_0)n}{M}} = 1,$ $\quad \sum_{n=0}^{M-1} e^{i2\pi\frac{(\mu-p_0)n}{M}} = M$

Solution of the equation of motion with wake fields

Otherwise,

$$\sum_{n=0}^{M-1} e^{i2\pi\frac{(\mu-p_0)n}{M}} = \frac{1-e^{i2\pi(\mu-p_0)}}{1-e^{i2\pi\frac{(\mu-p_0)}{M}}} = 0$$

$$\sum_{n=0}^{M-1} r^n = \frac{1-r^M}{1-r}$$

Here, μ and p_0 are integers.

$$e^{i2\pi(\mu-p_0)} = 1$$

Let $p \equiv -\frac{\mu-p_0}{M}$ $\quad p_0 = pM + \mu$

$$spectrum = \omega_0 \sum_{p_0=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i2\pi\frac{(\mu-p_0)n}{M}} \delta(\omega - \omega_\beta - p_0\omega_0)$$

$$= M\omega_0 \sum_{p=-\infty}^{\infty} \delta(\omega - \omega_\beta - pM\omega_0 - \mu\omega_0)$$

- To find the effect of the wake field, we have to evaluate the impedance at frequencies corresponding to frequencies present in the beam spectrum.

Physical interpretation of impedance

- Longitudinal wake function is defined $\Delta\delta(z) = -\frac{r_e}{\gamma} N(z')W_{\parallel}(z - z')$

- For the case of a charge distribution $\lambda(z)$ (number of particles per unit length):

$$\Delta\delta(z) = -\frac{r_e}{\gamma}\int \lambda(z')W_{\parallel}(z - z')dz'$$

- We write the longitudinal charge distribution as a mode decomposition

$$\lambda(z') = \frac{1}{2\pi\, C}\int \tilde{\lambda}(\omega)e^{i\frac{\omega z'}{c}}\, d\omega$$

- make the change of variables $z' \rightarrow z - z'$:

$$\Delta\delta(z) = \frac{1}{2\pi\, C}\frac{r_e}{\gamma}\int\int \tilde{\lambda}(\omega)e^{i\frac{\omega z}{c}}W_{\parallel}(z')e^{-i\frac{\omega z'}{c}}dz'd\omega$$

Physical interpretation of impedance

- We define the longitudinal impedance $$Z_{\parallel}(\omega) = \frac{Z_0 c}{4\pi}\int W_{\parallel}(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c}$$

- In terms of the impedance, the energy loss as function of longitudinal position z becomes

$$\Delta\delta(z) = \frac{c}{2\pi\, C}\frac{r_e}{\gamma}\frac{4\pi}{Z_0 c}\int \tilde{\lambda}(\omega) Z_{\parallel}(\omega) e^{i\frac{\omega z}{c}} d\omega$$

- Integrating with dz /c,

$$\int_{z=-\infty}^{z=\infty} \Delta\delta(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c} = \frac{c}{2\pi\, C}\frac{r_e}{\gamma}\frac{4\pi}{Z_0 c}\int_{z=-\infty}^{z=\infty}\int_{\omega'=-\infty}^{\omega'=\infty} \tilde{\lambda}(\omega') Z_{\parallel}(\omega') e^{i\frac{\omega' z}{c}} d\omega'\, e^{-i\frac{\omega z}{c}} \frac{dz}{c}$$

$$= \frac{c}{2\pi\, C}\frac{r_e}{\gamma}\frac{4\pi}{Z_0 c}\int_{\omega'=-\infty}^{\omega'=\infty} \tilde{\lambda}(\omega') Z_{\parallel}(\omega') d\omega' \int_{z=-\infty}^{z=\infty} e^{i(\omega'-\omega)\frac{z}{c}} \frac{dz}{c}$$

Physical interpretation of impedance

where $\int_{z=-\infty}^{z=\infty} e^{i(\omega'-\omega)\frac{z}{c}} \frac{dz}{c} = 2\pi\delta(\omega'-\omega)$

- Therefore, $\int_{z=-\infty}^{z=\infty} \Delta\delta(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c} = \frac{c}{2\pi\, C} \frac{r_e}{\gamma} \frac{4\pi}{Z_0 c} (2\pi)\, \tilde{\lambda}(\omega) Z_\parallel(\omega)$

- By applying following relations,

$$\Delta\delta(z) = \frac{\Delta E(z)}{E_0} = \frac{\Delta E(z)}{\gamma m_e c^2}, \qquad r_e = \left(\frac{Z_0 c}{4\pi}\right) \frac{e^2}{m_e c^2},$$

- We can get $\int \frac{\Delta E(z)}{e} e^{-i\frac{\omega z}{c}} \frac{dz}{c} = \frac{1}{C} ec\tilde{\lambda}(\omega) Z_\parallel(\omega)$

Physical interpretation of impedance

- Left hand side is the Fourier transform of a voltage (the energy change of a particle over one turn of the accelerator). Right hand side is the product of the current spectrum and the impedance.

$$\int \frac{\Delta E(z)}{e} e^{-i\frac{\omega z}{c}} \frac{dz}{c} = \frac{1}{C} ec\tilde{\lambda}(\omega) Z_{\parallel}(\omega)$$

- In other words, the impedance — defined as the Fourier transform of the Wake function — relates the voltage seen by the beam (resulting from the interaction of the beam with its surroundings) to the beam current in frequency space.

$$\tilde{V}(\omega) = \tilde{I}(\omega) Z_{\parallel}(\omega)$$

Example: coupled bunch motion with resistive-wall wake field

- Consider the case of the transverse resistive-wall wake fields.

$$W_\perp(z) = -\frac{2}{\pi b^3}\sqrt{\frac{4\pi}{Z_0 c}\frac{c}{\sigma}}\frac{L}{\sqrt{-z}} \quad (z < 0) \qquad \frac{Z_\perp(\omega)}{L} \approx \frac{c}{\omega}\frac{1 - i\,\mathrm{sgn}(\omega)}{\pi b^3 \delta_{skin}\sigma}$$

- The frequency shift is given by:

$$\delta_{skin} = \sqrt{\frac{4\pi}{Z_0 c}\frac{c^2}{2\pi\sigma|\omega|}}$$

$$\Omega_\mu - \omega_\beta \approx -i\frac{4\pi}{Z_0 c}\frac{MN_0 r_e}{4\pi\gamma\nu_\beta}\frac{c}{T_0}\sum_{p=-\infty}^{\infty} Z_\perp[(pM + \mu)\omega_0 + \omega_\beta]$$

  - Resistive-wall impedance is proportional to $1/\sqrt{\omega}$
  - We expect to see the strongest effects in modes for which:

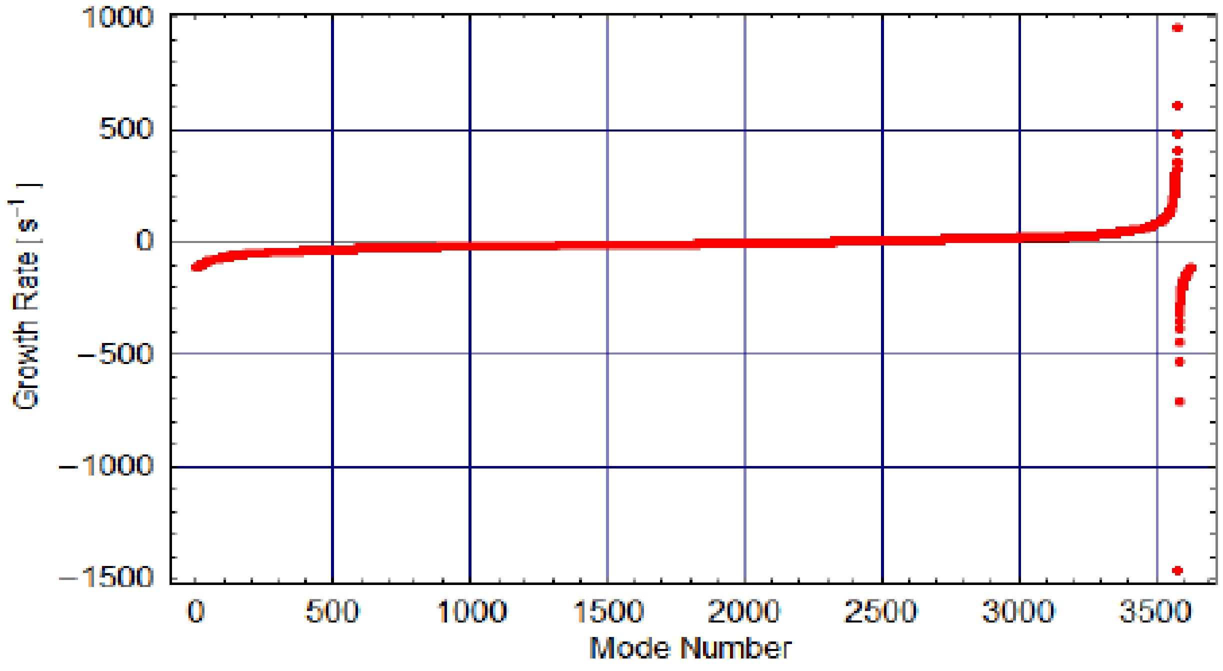
$$(pM + \mu)\omega_0 + \omega_\beta \approx 0 \rightarrow \mu \approx -pM - \nu_\beta \qquad (0 \le \mu < M, \qquad \nu_\beta > 0)$$

  - Therefore, the nearest mode to zero (most strongest mode) is when $p =$ -1.

$$\mu \approx M - \nu_\beta$$

Example: coupled bunch motion with resistive-wall wake field (ILC damping ring)

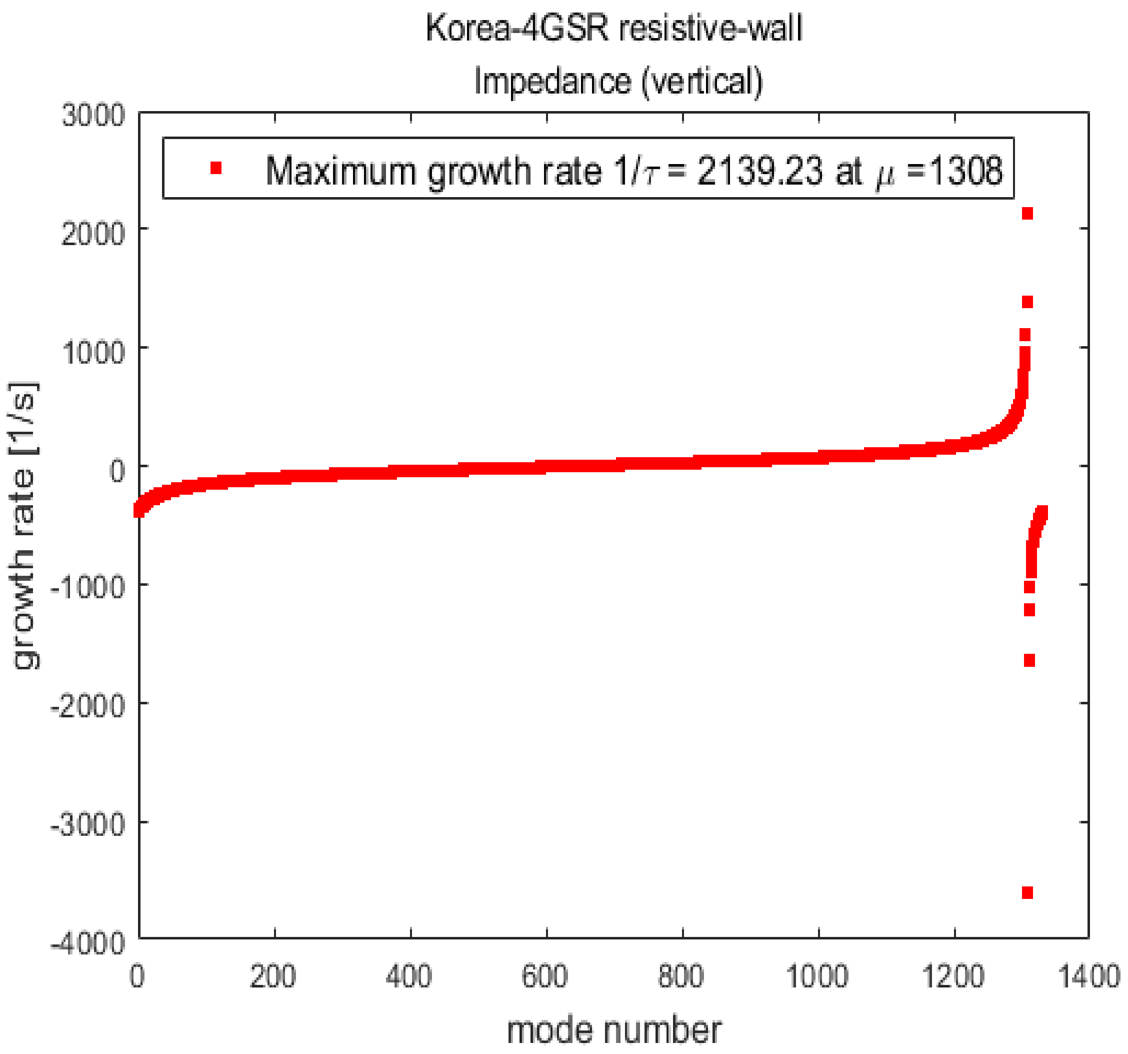
Ring parameters	ILC Damping ring
Beam Energy E_0	5 GeV
Circumference C	15.935 km
Average Current I_0	170 mA
The number of RF buckets	3455
Momentum Compaction α_c	0.474e-3
Revolution angular frequency ω_0	1.1821e5 rad/s
Tune $\nu_{x,y,s}$	75.783 / 76.413 / 0.252
Beam pipe radius b	32.9 mm
Electric conductivity of Beam pipe σ_c	3.8e7 S/m
RF frequency f_0	650 MHz
RF voltage V_{rf}	115 MV



(growth rate *vs* mode number for the resistive wall wake) Roughly half the modes grow (are unstable), and half the modes are damped.

The strongest effects are for mode numbers around $M - \nu_\beta$, (M is the number of bunches and ν_β is the tune)

Example: coupled bunch motion with resistive-wall wake field (4GSR ring case)



Ring parameters	Korea-4GSR
Beam Energy E_0	4 GeV
Circumference C	799.3 m
Average Current I_0	400 mA
RF harmonics h	1332
Momentum Compaction α_c	7.775e-5
RF frequency f_0	499.593 MHz
RF voltage V_{rf}	3.5 MV
Bunch length σ_τ	9.33 ps
Tune $\nu_{x,y,s}$	68.179 / 23.260 / 0.0034
Average Beta-function $\overline{\beta}_{x,y}$	4.642 / 9.959 m
Beam pipe radius b	9 mm
Electric conductivity of Beam pipe σ_c	3.01e7 S/m
Radiation damping time $\tau_{x,y,s}$	10.55 / 19.43 / 16.79 ms

Example: coupled bunch motion with resistive-wall wake field

- The frequency shift

$$\Omega_\mu - \omega_\beta \approx -i \frac{4\pi}{Z_0 c} \frac{M N_0 r_e}{4\pi\gamma\nu_\beta} \frac{c}{T_0} \sum_{p=-\infty}^{\infty} Z_\perp[(pM+\mu)\omega_0 + \omega_\beta]$$

$\mathrm{Im}(\Omega_\mu - \omega_\beta) > 0$ means instability growth by $Re(Z) < 0.$

If we include the largest term in the summation, we get the fastest growth rate of any of the modes that minimizes $\omega = (pM + \mu + \nu_\beta)\omega_0 \approx 0$

($M > \nu_\beta$, p=-1, $0 \le \mu < M$, μ = M-int(ν_β)-1)

Betatron tune has a fractional part, $\nu_\beta = N_\beta + \Delta_\beta$ $(0 < \Delta_\beta < 1)$. p, M, μ, N_β are integers.

Since the impedance is largest at low frequencies, the beam mode with the highest growth rate will be the mode with the the smallest negative value of ω : a negative value of ω will mean that the real part of $Z_\perp(\omega)$ will be negative, which will mean that the imaginary part of $\Omega_\mu - \omega_\beta$ will be positive.

Example: coupled bunch motion with resistive-wall wake field

Substituting $\omega = (\Delta_\beta - 1)\omega_0$ into resistive-wall impedance,

$$\frac{Z_\perp\left((\Delta_\beta - 1)\omega_0\right)}{C} = \frac{c}{(\Delta_\beta - 1)\omega_0}\frac{1+i}{\pi b^3\sigma}\sqrt{\frac{Z_0 c}{4\pi}\frac{2\pi\sigma(1-\Delta_\beta)\omega_0}{c^2}}$$

Growth rate for fastest−growing mode

$$\Gamma \approx \frac{4\pi}{Z_0 c}\frac{C}{2\pi b^3}\frac{c}{\gamma}\frac{< I >}{I_A}\frac{1}{2\pi\nu_\beta}\sqrt{\frac{Z_0 c}{4\pi}\frac{c}{\sigma}}\sqrt{\frac{C}{1-\Delta_\beta}}$$

$< I >$ is the average current (=$MN_0\, ec$ /C) $I_A = \frac{ec}{r_e} \approx$ 17.045 kA

Bunch-by-bunch feedback systems

- Parameters that determine the damping rate from feedback system are:
  - beta functions at the pick-up (s_1) and the kicker (s_2)
  - betatron phase advance between the pick-up and the kicker
  - amplifier gain, g defined by: $\Delta p_y(s_2) = g y(s_1)$

  $y(s_1)$ is bunch position at the pick-up (at location s_1), and $\Delta p_y(s_2)$ is kick applied to the bunch by the kicker (at location s_2).

- In terms of the action J and angle ϕ variables, transverse coordinate and momentum of a particle at the pick-up can be

$$y_1 = \sqrt{2\beta_1 J_1}\cos(\phi_1) \qquad p_{y1} = -\sqrt{\frac{2J_1}{\beta_1}}(\sin(\phi_1) + \alpha_1\cos(\phi_1))$$

Bunch-by-bunch feedback systems

Following the kicker, the coordinate and momentum

$$y_2 = \sqrt{2\beta_2 J_1}\cos(\phi_1 + \Delta\phi_{21})$$

$$p_{y2} = -\sqrt{\frac{2J_1}{\beta_2}}[\sin(\phi_1 + \Delta\phi_{21}) + \alpha_2 \cos(\phi_1 + \Delta\phi_{21})] + g y_1$$

We can write in terms of a new action and angle:

$$y_2 = \sqrt{2\beta_2 J_2}\cos(\phi_2)$$

$$p_{y2} = -\sqrt{\frac{2J_2}{\beta_2}}(\sin(\phi_2) + \alpha_2 \cos(\phi_2))$$

Bunch-by-bunch feedback systems

We can rewrite coordinate and momentum as follows

$$y_2 = \sqrt{2\beta_2 J_1}\cos(\phi_1 + \Delta\phi_{21}) = \sqrt{2\beta_2 J_2}\cos(\phi_2)$$

$$\rightarrow \sqrt{J_1}\cos(\phi_1 + \Delta\phi_{21}) = \sqrt{J_2}\cos(\phi_2) \qquad (1)$$

$$p_{y2} = -\sqrt{\frac{2J_2}{\beta_2}}(\sin(\phi_2) + \alpha_2\cos(\phi_2)) = -\sqrt{\frac{2J_1}{\beta_2}}[\sin(\phi_1 + \Delta\phi_{21}) + \alpha_2\cos(\phi_1 + \Delta\phi_{21})] + gy_1$$

$$\rightarrow \left(\sqrt{J_2}\sin(\phi_2) - \sqrt{J_1}\sin(\phi_1 + \Delta\phi_{21})\right) + \alpha_2\left(\sqrt{J_2}\cos(\phi_2) - \sqrt{J_1}\cos(\phi_1 + \Delta\phi_{21})\right) = -\sqrt{\frac{\beta_2}{2}}gy_1$$

by (1)

where, $y_1 = \sqrt{2\beta_1 J_1}\cos(\phi_1)$ $\rightarrow \sqrt{J_2}\sin(\phi_2) = \sqrt{J_1}[\sin(\phi_1 + \Delta\phi_{21}) - \sqrt{\beta_1\beta_2}g\cos(\phi_1)]$ (2)

Using square of Eq(1) and Eq.(2),

$$\rightarrow J_2 = J_1[1 - 2g\sqrt{\beta_1\beta_2}\cos\phi_1\sin(\phi_1 + \Delta\phi_{21}) + g^2\beta_1\beta_2\cos^2(\phi_1)]$$

Bunch-by-bunch feedback systems

Averaging over the initial phase angle ϕ_1, $J_2 = J_1\left[1 - g\sqrt{\beta_1\beta_2}\,\sin(\Delta\phi_{21}) + \frac{1}{2}g^2\beta_1\beta_2\right]$

If the phase advance $\Delta\phi_{21}$ is close to the (optimal) value of 90°,
the new action can be rewritten by Taylor expansion of exp-function.

$$e^{-x} \approx 1 - x + \frac{1}{2}x^2, \qquad x = g\sqrt{\beta_1\beta_2}\,\sin(\Delta\phi_{21}) \ll 1,$$

$$J_2 \approx J_1 \exp\left(-g\sqrt{\beta_1\beta_2}\sin(\Delta\phi_{21})\right) \approx J_1 \exp(-\frac{2T_0}{\tau_{FB}})$$

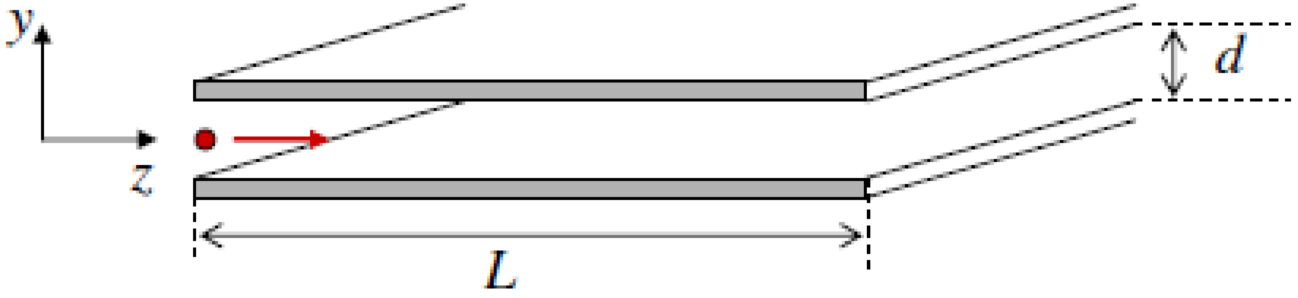
where τ_{FB} is the damping time of the feedback system

$$\frac{1}{\tau_{FB}} = \frac{g\sqrt{\beta_1\beta_2}\sin\Delta\phi_{21}}{2T_0}$$

Bunch-by-bunch feedback systems

- From the required damping rate, we can calculate the required gain for the feedback system. The gain of the feedback system determines the voltage applied to the kicker.

- Consider a kicker consisting of two infinitely wide parallel plates of length L, separated by a distance d and with a voltage V between them.



The deflection (change in normalized momentum) of the bunch from passing between the plates

$$\Delta p_{y1} = \frac{F_x}{P_o}\frac{L}{c} = 2\frac{V}{E/e}\frac{L}{d}$$

Bunch-by-bunch feedback systems

- The kicker voltage per unit bunch offset is given by $\frac{dV}{dy} = \frac{1}{2}\frac{E}{e}\frac{d}{L}g$

- Consider a feedback system used to damp the resistive wall instability in the ILC damping rings. If we assume a maximum growth time of 40 turns, beta functions of 10 m at the pick-up and kicker, and a phase advance of 90° between them, the required gain for the feedback system is:

$$g = 2\frac{T_0}{\tau_{FB}}\frac{1}{\sqrt{\beta_1\beta_2}} = 0.005$$

- If we assume kickers of length 20 cm and separated by 2 cm, and a 5 GeV beam, the kicker voltage per unit bunch offset at the pick-up is:

$$\frac{dV}{dy} = \frac{1}{2}\frac{E}{e}\frac{d}{L}g = 1.25 \text{ kV/mm}$$

Bunch-by-bunch feedback systems

- We consider the effect of noise on the pick-up, or in the amplifier. This will lead to some variation in the applied kick from the "correct" value; which will result in some excitation of betatron motion.
- Let us represent the noise in the feedback system by the addition of a quantity δ_y to the bunch position measured by the pick-up: $y_1 \rightarrow y_1 + \delta y$

$$(y_1 = \sqrt{2\beta_1 J_1}\cos(\phi_1) + \delta y)$$

averaging with ϕ_1,

$$J_2 = J_1\left[1 - g\sqrt{\beta_1\beta_2}\sin(\Delta\phi_{21}) + \frac{1}{2}g^2\beta_1\beta_2\right] + \frac{\beta_2}{2}g^2(\delta y)^2$$

$$J_2 \approx J_1 \exp\left(-\frac{2t}{\tau_{tot}}\right) + \frac{1}{2}g^2\beta_2(\delta y)^2$$

This will modify the change in the action resulting from the voltage applied to the kicker

Bunch-by-bunch feedback systems

- Including the effect of noise in the feedback system, we can write the equation of motion for the action as:

$$\frac{dJ_2}{dt} \approx \frac{g^2\beta_2 < \delta y^2 >}{2T_0} - \frac{2}{\tau_{tot}} J_1$$

We see that the action reaches an equilibrium with $J_1 \approx J_2$:

$$\frac{dJ_2}{dt} = -\frac{2J_2^{eq}}{\tau_{tot}} \exp\left(-\frac{2T_0}{\tau_{tot}}\right) + \frac{g^2\beta_2 < \delta y >^2}{2T_0} = 0$$

$$J_{equ} \approx \frac{\tau_{tot}}{4T_0} g^2\beta_2 < \delta y^2 >$$

Bunch-by-bunch feedback systems

- Let us assume that we double the gain of the feedback system, compared to that required to exactly balance the resistive-wall instability, so that:

$$g = 4\frac{T_0}{\tau_{RW}}\frac{1}{\sqrt{\beta_1\beta_2}}, \qquad J_{equ} \approx 4\frac{T_0}{\tau_{RW}}\frac{<\delta y^2>}{\beta_1}$$

- Let us also assume that the specification on the bunch-to-bunch beam jitter is a fraction f of the beam size:

$$2J_{equ} < f^2\epsilon_y \qquad ({y^2}_m = 2\beta J, \quad (f\sigma_y)^2 > 2\beta J, \quad f^2\beta\epsilon_y > 2\beta J)$$

- This sets an upper limit on the feedback system noise:

$$<\delta y^2> < \frac{f^2}{8}\frac{\tau_{RW}}{T_0}\beta_1\epsilon_y$$

Bunch-by-bunch feedback systems

- As an example, consider the ILC damping rings. Let us assume that f = 10%, the beta function at the pick-up is 10 m, that the resistive-wall growth time is 40 turns, and that the equilibrium vertical emittance is 2 pm.

- In other words, the pick-up needs a resolution of better than 1μm (neglecting any additional noise from the amplifier).

$$\sqrt{< \delta y^2 >} \quad < 1\, \mu m$$

This is a challenging, but not unrealistic specification.

Summary

- Long-range wake fields couple the motion of different bunches in a storage ring. Depending on a range of factors (including the characteristics of the wake fields, the beam current, beam energy, synchrotron radiation damping rates etc.) this can lead to instabilities, in which the oscillations of the bunches grow exponentially.

- Sources of long-range wake fields in storage rings include the finite resistance of the vacuum chamber walls, and higher-order modes in the RF cavities (and, possibly, other components).

- A simple analytical model of the long-range wake field effects leads to an estimate of the growth rates in terms of the impedance.

- Coupled-bunch instabilities can be controlled using bunch-by-bunch feedback systems. For the ILC damping rings, bunch-to-bunch jitter excited by noise in the feedback system (pick-up or amplifier) is a concern.