Fast-Ion Instability

Reference

- Andy Wolski, Head-tail instabilities

- A. Chao, Physics of collective beam instabilities in high energy accelerator, 1993

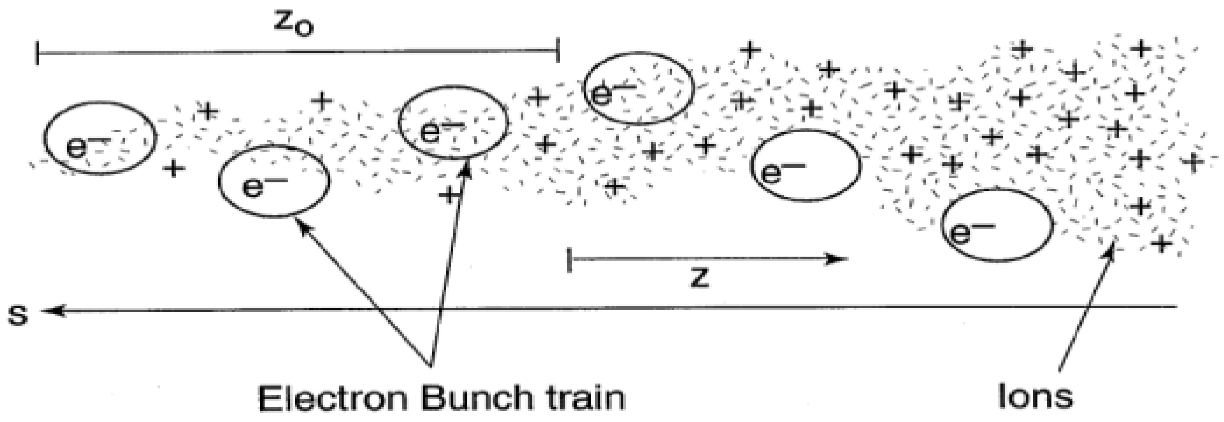
Fast-ion instability

■ In electron storage rings, residual gas molecules can be ionized by the beam. The resulting positive ions are trapped in the electric field of the beam, and accumulate to high density.
→ The fields of the ions can then drive beam instabilities.

■ While electrons move rapidly on the time scale of a single bunch passage, ions move relatively slowly. The dynamical behaviour is then somewhat different.

■ If a storage ring is uniformly filled with electron bunches, **then ions accumulate over many turns**. This leads to the well-known phenomenon of ion trapping, which is usually solved by including "gaps" in the fill pattern.

■ Under certain conditions, sufficient ions can accumulate in the passage of a small number of bunches to drive an instability, known as the "fast ion instability".



Fast-ion instability

Equation of motion of an ion due to electric field of electron bunch

$$M_{ion}\,\ddot{y}_i = q_i\,E_y^{elec} \quad , \quad E_y^{elec} = \frac{\lambda\, y}{2\pi\varepsilon_0\sigma_y(\sigma_x+\sigma_y)}$$

$$\ddot{y}_i = -\frac{e\lambda y_i}{Am_p\,2\pi\varepsilon_0\sigma_y(\sigma_x+\sigma_y)}$$

$$\ddot{y}_i = \frac{-2c^2(e^2)\lambda\, y_i}{Ae(4\pi\varepsilon_0 m_p c^2)\sigma_y(\sigma_x+\sigma_y)} = \frac{-\lambda\, 2c^2\, r_p y_i}{eA\sigma_y(\sigma_x+\sigma_y)}$$

$$\left(\lambda = \frac{eN_b}{l_b}, \qquad r_p = \frac{e^2}{4\pi\varepsilon_0\, m_p c^2}\ \text{: classical proton radius}\right)$$

$$= \frac{-N_b}{l_b}\frac{2c^2 r_p y_i}{A\sigma_y(\sigma_x+\sigma_y)}$$

$$\dot{y}_i = \int \ddot{y}_i dt\,, \qquad dt = \frac{l_b}{c}$$

$$\dot{y}_i = -\frac{2N_b r_p c\ \ y_i}{A\sigma_y(\sigma_x+\sigma_y)} + \dot{y}_i(0) \qquad\qquad k = \frac{2N_b r_p c}{A\sigma_y(\sigma_x+\sigma_y)}$$

n : number of bunches per beam, l_b : bunch length A : atomic mass of ion

N_b : number of electron / bunch q_i : charge of ion M_{ion}: mass of ion

Fast-ion instability

Mechanism of ion trapping in bunch can be modelled as the ions experience a focusing force from a bunch and drift in bunch gaps.

Vertical displacement of trapped ion at a time t and a position s is modelled as

$$\begin{pmatrix} y \\ \dot{y} \end{pmatrix}_{t=t} = \begin{pmatrix} 1 & t_b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix}_{t=0}$$

t_b : bunch separation, linear kick : $k = \frac{2N_b r_p c}{A\sigma_y(\sigma_x+\sigma_y)}$

Ion motion remains stable when -2 ≤ Tr(M) ≤ 2

$$M = \begin{pmatrix} 1 - t_b k & t_b \\ -k & 1 \end{pmatrix}, \quad -2 \le 2 - t_b k \le 2, \quad 0 \le t_b k \le 4$$

$$\frac{2N_b r_p c t_b}{A\sigma_y(\sigma_x + \sigma_y)} \le 4,$$ It is satisfied if $A \ge A_c$

Critical mass $A_c = \frac{N_b r_p c t_b}{2\sigma_y(\sigma_x+\sigma_y)}$: This condition defines a critical mass that ions performs stable oscillation and trapped in electron beam when their atomic mass exceeds A_c.

A_c is the minimum atomic mass of ions that can be trapped.

Fast-ion instability

Ion production rate $\boldsymbol{n_i(m^{-1})}$ created by a bunch with population $\boldsymbol{N_e}$ is

$$n_i = d_m \sigma_m N_e$$

molecular density $d_m(m^{-3}) = \frac{P}{kT} = 2.42 \times 10^{20}\ P\ [pascal]$

$$\sigma_{CO} = 1.9 \times 10^{-22} m^2$$

$$\sigma_{H_2} = 3.2 \times 10^{-32} m^2$$

Ion line density at the tail of bunch fraction is

$$\lambda_i = \sigma_{ion} \frac{P}{kT} N_e n_b$$

$(N_e, n_b :\ number\ of\ electrons\ in\ a\ bunch\ and\ number\ of\ bunches)$

$$\lambda_i = 0.045 N_e n_b P\ [pascal]$$

$$= 6 N_e n_b\ P\ [Torr] \quad \text{for CO}$$

effective ion density : $\rho_{i\,eff} = \frac{\lambda_i}{\frac{3}{2}\sigma_y(\sigma_x + \sigma_y)}$

Fast-ion instability

Estimation of wake field amplitude of ion cloud

Wake force of ion cloud generated from a displaced electron bunch by Δy is given by transverse kick per unit of both charges per unit of transverse displacement

$$W_y = \frac{\int F ds}{q_i q_e \Delta y} = \frac{\int \frac{dp}{dt} ds}{q_i q_e \Delta y} = \frac{c\Delta p}{q_i q_e \Delta y}$$

$$q_e = N_e e, \quad q_i = e$$

$$\Delta p = m_e \Delta v$$

$$\Delta y' = \frac{\Delta y}{\Delta s} = \frac{\Delta y}{\gamma c \Delta t} = \frac{\Delta v}{\gamma c}$$

$$= \frac{c\Delta p}{N_e r_e m_e c^2 \Delta y}$$

$$= \frac{\Delta v}{N_e r_e c\, \Delta y} \qquad (\Delta v : velocity\ kick\ by\ electron\ bunch\ from\ ion\ cloud)$$

$$= \frac{\gamma}{N_e\, r_e} \frac{\Delta y\prime}{\Delta y}$$

Equation of motion for electron

$$m_e \ddot{y}_e = e E_y^{ion} \quad , \quad E_y = \frac{\lambda\, y}{2\pi\varepsilon_0\, \sigma_y(\sigma_x + \sigma_y)}$$

Fast-ion instability

$$\ddot{y}_e = \frac{e\,\lambda_i\, y_i}{2\pi\varepsilon_0 m_e\, \sigma_y(\sigma_x+\sigma_y)} \qquad \left(\sigma_y = \sqrt{\sigma_{yi}^2 + \sigma_{ye}^2}\ ,\ \sigma_{yi} = \frac{\sigma_{ye}}{\sqrt{2}}\ ,\ \sigma_y = \sqrt{\frac{3}{2}}\sigma_y\ ,\quad \sigma_x = \sqrt{\frac{3}{2}}\sigma_x\right)$$

$$\dot{y}_e = \int \frac{e\,\lambda_i\, y_i}{2\pi\varepsilon_0 m_e \frac{3}{2}\sigma_y(\sigma_x+\sigma_y)} dt$$

$$= \int \frac{e\,\lambda_i\, 2c^2}{4\pi\varepsilon_0 m_e c^2} \frac{dt\; y_i}{\frac{3}{2}\sigma_y(\sigma_x+\sigma_y)} \qquad \left(\lambda_i = \frac{eN_i}{l_{sep}}\right)$$

$$= \int \frac{r_e\,\lambda_i\, 2c^2\, \frac{l_{sep}}{c}\, y_i}{e\, \frac{3}{2}\, \sigma_y(\sigma_x+\sigma_y)} \qquad \left(dt = \frac{l_{sep}}{c}\right)$$

$$= \frac{2r_e c\, N_i\, y_i}{\frac{3}{2}\sigma_y(\sigma_x+\sigma_y)}$$

$$\dot{y}_e = \Delta v = \frac{2r_e c N_i y_i}{\frac{3}{2}\sigma_y(\sigma_x+\sigma_y)}$$

Fast-ion instability

Equation of motion for ion is

$$M_{ion}\,\ddot{y}_i = q_i E_y^{elec} \qquad \dot{y}_i = \Delta v_{yi} = \frac{2r_p\, c\, N_e\, \Delta y}{A\left(\frac{3}{2}\right)\sigma_y(\sigma_x+\sigma_y)}$$

$$y_i = \frac{\Delta v_{yi}}{\omega_i}$$

$$\omega_i = c\left(\frac{4N_e r_p}{3L_{sep}\sigma_y\left(\sigma_x+\sigma_y\right)A}\right)^{\frac{1}{2}}$$

Amplitude of wake function is

$$W_y = \frac{\gamma}{N_e r_e}\frac{\Delta y\prime}{\Delta y} \qquad \left(\Delta v = \frac{2r_e c N_i y_i}{\frac{3}{2}\sigma_y(\sigma_x+\sigma_y)},\quad \Delta y' = \frac{\Delta v}{\gamma c}\right)$$

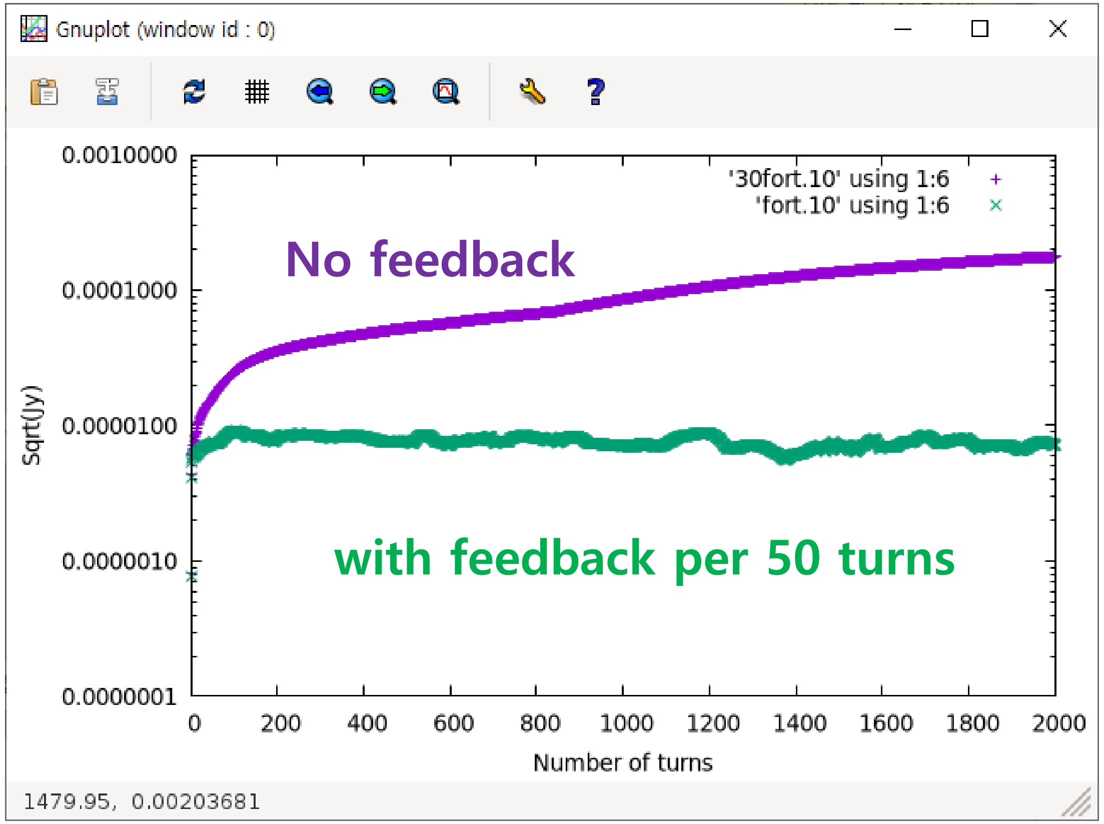
$$= \frac{\gamma}{N_e r_e} * \frac{2\gamma_e c N_i y_i}{\frac{3}{2}\sigma_y\left(\sigma_x+\sigma_y\right)} * \frac{1}{\gamma c} * \frac{1}{\Delta y}$$

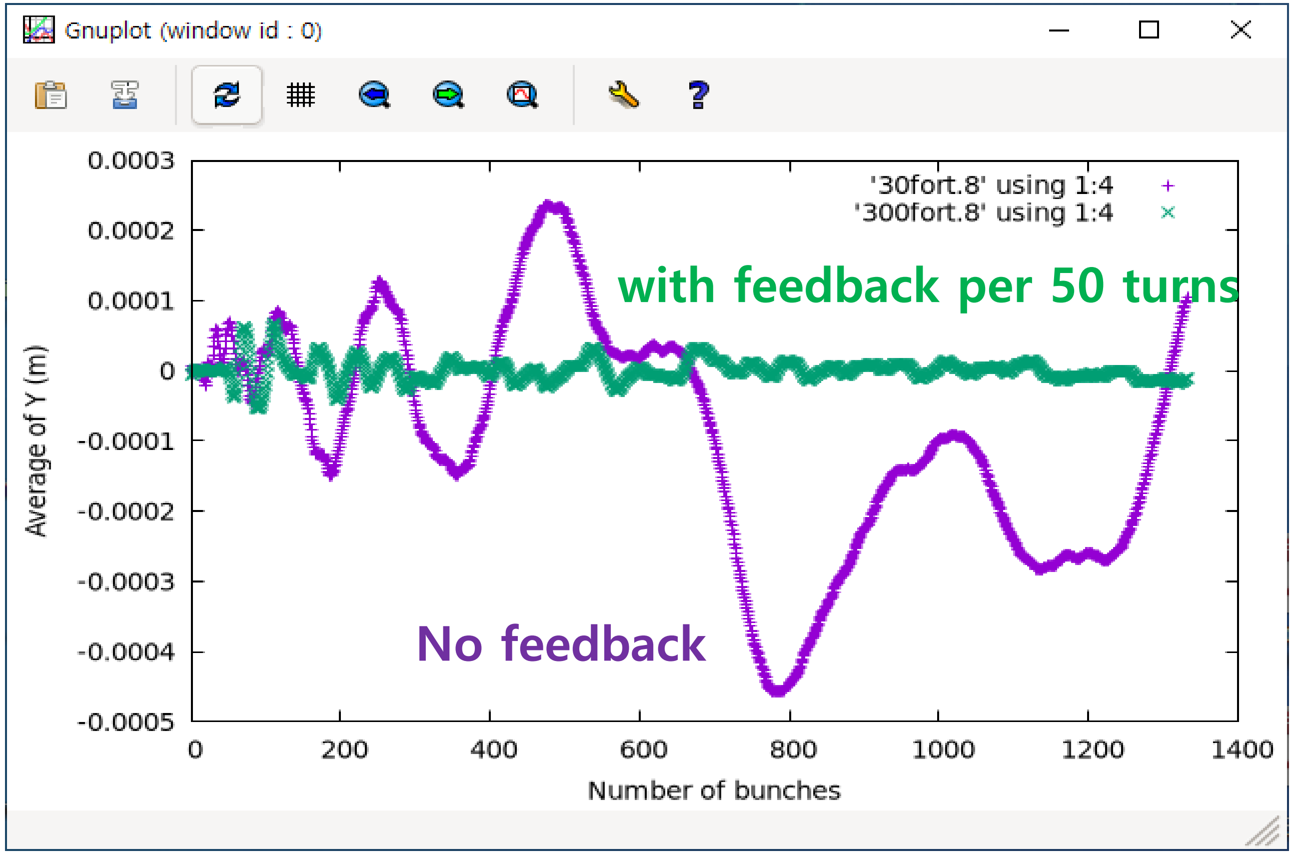
$$= \frac{\gamma}{N_e \gamma_e} * \frac{2\gamma_e c N_i}{\frac{3}{2}\sigma_y\left(\sigma_x+\sigma_y\right)} * \frac{1}{\gamma c} * \frac{2r_p c N_e\ \ \Delta y}{A\frac{3}{2}\sigma_y\left(\sigma_x+\sigma_y\right)\ \ \Delta y} * \frac{1}{c\left(\frac{4N_e r_p}{3L_{sep}\sigma_y\left(\sigma_x+\sigma_y\right)A}\right)^{\frac{1}{2}}}$$

$$= N_i\left(\frac{4}{3}\frac{1}{\sigma_y\left(\sigma_x+\sigma_y\right)}\right)^{\frac{3}{2}}\sqrt{\frac{L_{sep} r_p}{A N_e}}$$

Example : Fast-ion instability

1332 bunches 400 mA 0.1 nT

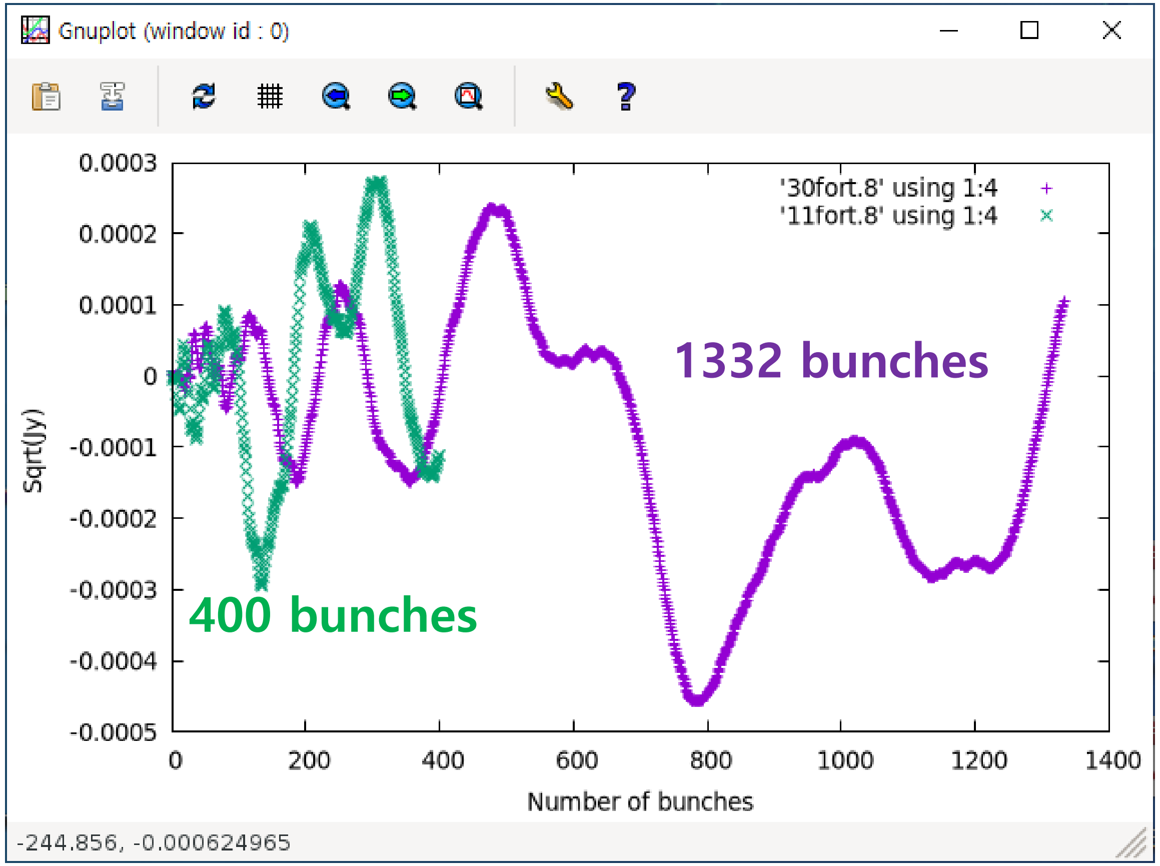




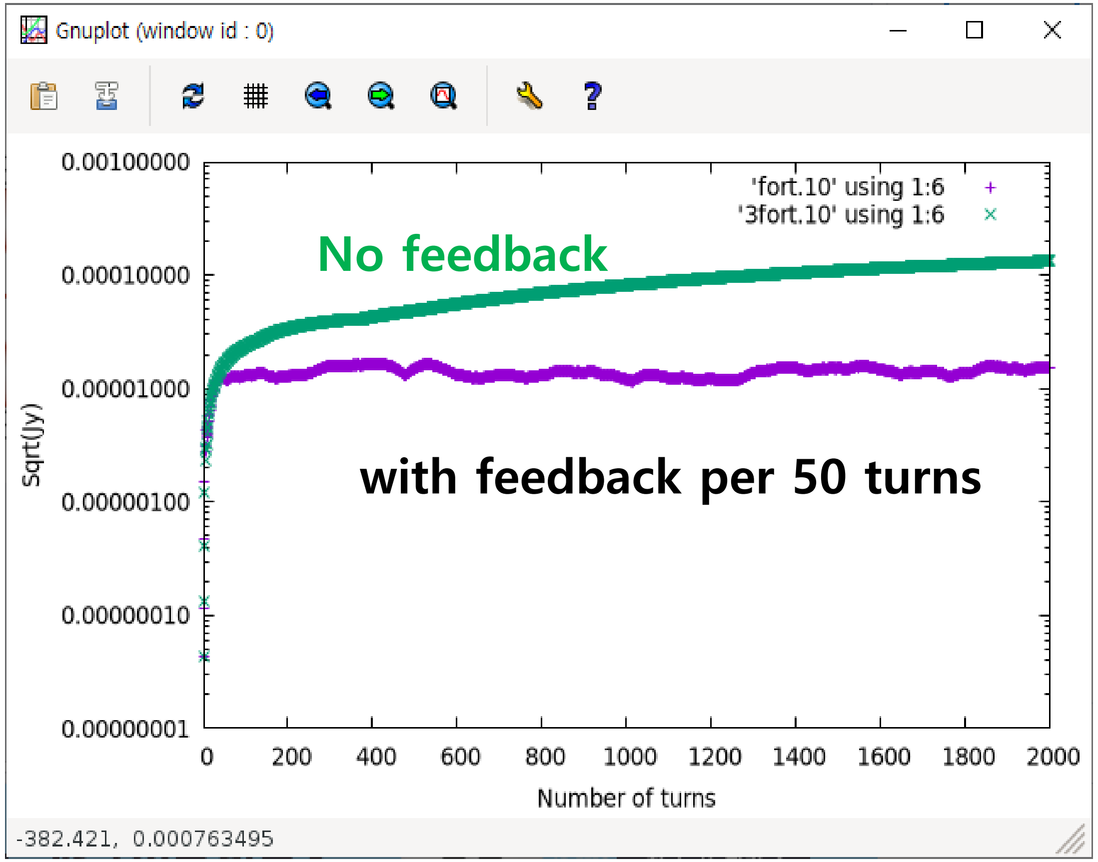
Example : Fast-ion instability

400 mA 0.1 nT

1 train (1332 and 400 bunches)



100 bunches with 4 trains



Strong Head Tail Instability

The macroparticle model

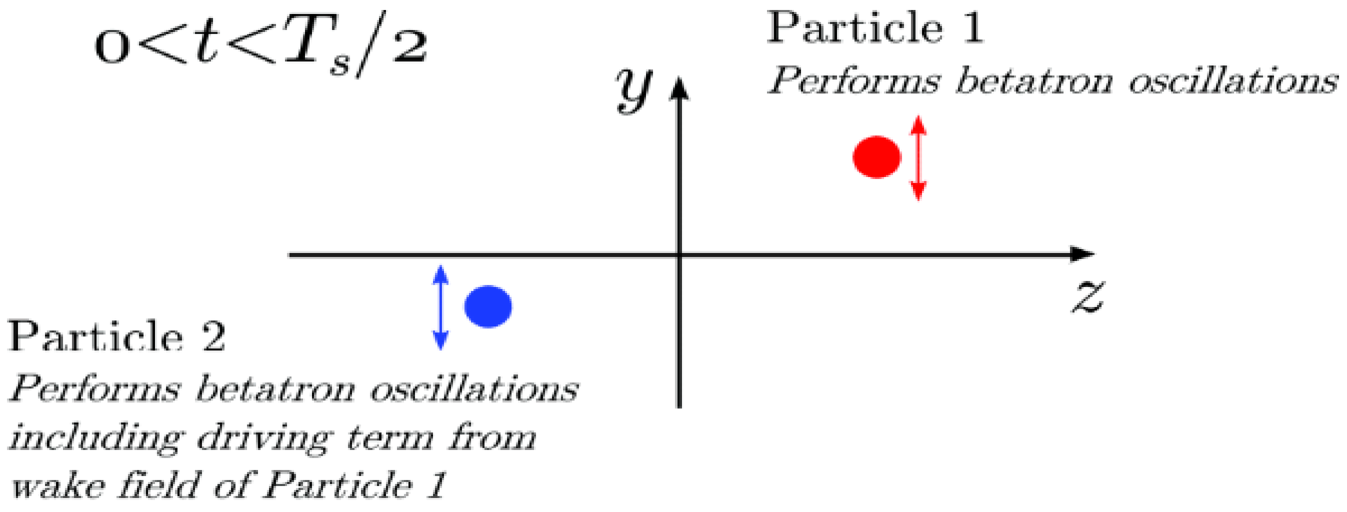
- Some of the important aspects of the head-tail instability can be understood in terms of a simple model of a bunch as consisting of just two particles, each with charge equal to half the total bunch charge.

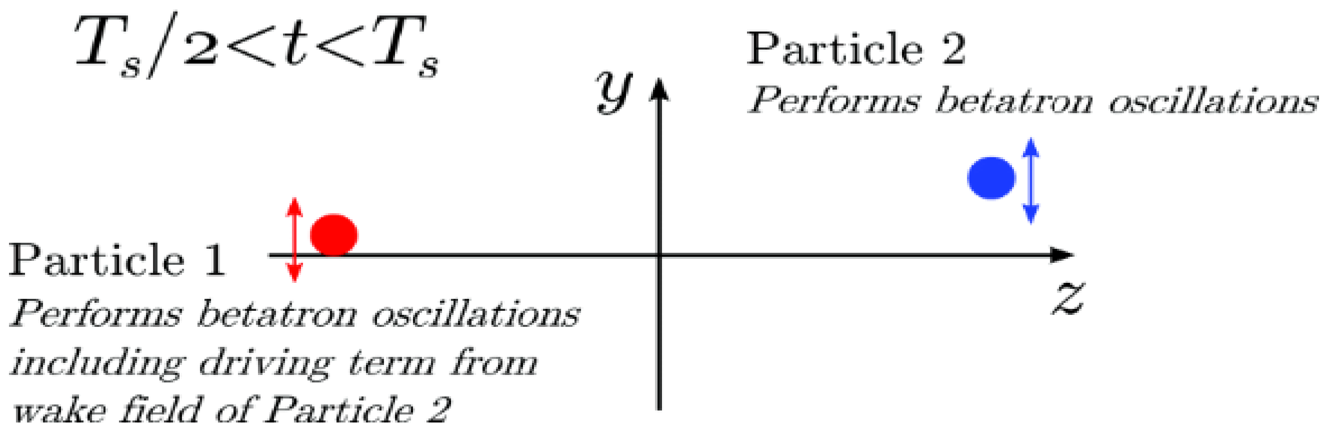
- We shall base our analysis of the head-tail instability on this model of a bunch consisting of two macroparticles.

Transverse motion with wake fields: equations of motion

- Consider a “bunch” consisting of two particles in a storage ring, with particle 1 ahead of particle 2 by a distance of order of the bunch length σ_z.
- We assume that the particles are ultra-relativistic. Both particles will perform betatron oscillations as they travel around the ring.
- In the absence of wake fields, the betatron frequency is determined solely by the focusing in the lattice.
- When wake fields are present, the trailing particle will experience additional forces from the wake field generated by the leading particle. Synchrotron motion means that the leading and trailing particles interchange roles after half a synchrotron period, T_s.

Transverse motion with wake fields: equations of motion





Transverse motion with wake fields: equations of motion

- Because the particles are ultra-relativistic, the motion of particle 1 will not be affected by any wake fields from particle 2. The equation of motion for transverse oscillations of particle 1 can be written :

$$\ddot{y}_1 + \omega_\beta^2 y_1 = 0$$

where the dots indicate second derivative with respect to time, and ω_β is the betatron oscillation frequency.

Transverse motion with wake fields: equations of motion

- Particle 2 will observe the wake field from particle 1. When the particles pass through a section with wake function $W_\perp(-\Delta z)$, the transverse deflection of particle 2 from the wake field will be:

$$\Delta p_2 = -\frac{e^2 N_0}{2E_0} y_1 W_\perp(-\Delta z)$$

each particle has total charge $eN_0/2$ and total energy $E_0\, N_0\, /2$.

y_1 is the transverse co-ordinate of particle 1, and the particles have longitudinal separation Δz.

- The deflection can be included as a "driving force" in the equation of motion for particle 2. Taking into account the usual betatron oscillation, the equation of motion is

$$\ddot{y}_2 + \omega_\beta^2 y_2 = -\frac{c^2 e^2 N_0}{2E_0} y_1 \frac{W_\perp(-\Delta z)}{L}$$

where L is the length of the accelerator with wake function $W_\perp(-\Delta z)$

$$\left(= -\frac{c^2\, F_y}{E_0\, N_2} = -\frac{c^2\, M_d}{E_0\, N_2\, L}\right), \qquad (M_d = y_1\, (e\, N_0/2)^2\, W_\perp(-\Delta z)$$

Transverse motion with wake fields: equations of motion

- We can make a linear approximation for the wake function, so that

$$W_{\perp}(-\Delta z) = \frac{\Delta z}{\sigma_z} W_0$$

W_0 is a constant and σ_z is bunch length. The wake function per unit length is $W_0\,\Delta z/C_0\,\sigma_z$, where C_0 is the circumference of the ring. We assume that the beta function is constant, so that the betatron frequency ω_β is also constant. The equations of motion are:

$$\ddot{y}_1 + \omega_\beta^2 y_1 = 0$$
$$\ddot{y}_2 + \omega_\beta^2 y_2 = -iAy_1 e^{-i\omega_s t}$$

constant A is defined $$A = \frac{c^2 e^2 N_0}{2E_0}\frac{W_0}{C_0}$$

We have taken the synchrotron motion into account by writing: $\Delta z = i\sigma_z e^{-i\omega_s t}$, where ω_s is the synchrotron frequency. The particles start with zero separation at t= 0.

Solution to the equations of motion

- For particle 1, the solution is straightforward $y_1(t) = y_1(0)\mathrm{e}^{-i\omega_\beta t}$

- Particle 2 has a natural oscillation at frequency ω_β, but is also subject to a driving force at frequency $\omega_\beta + \omega_s$. Thus, we write a solution to the equation of motion

$$y_2(t) = B_1 e^{-i\omega_\beta t} + B_2 e^{-i(\omega_\beta+\omega_s)t}$$

- The constants B_1 and B_2 can be determined from the initial condition $B_1 + B_2 = y_2(0)$, and by substituting the solution into the equation of motion. Assuming that $\omega_s \ll \omega_\beta$

$$B_2 \approx \frac{iAy_!(0)}{2\omega_\beta\omega_s}, \qquad B_1 \approx y_2(0) - \frac{iAy_1(0)}{2\omega_\beta\omega_S}$$

We find from the solution to the equations of motion:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \approx e^{-i\omega_\beta T_S/2} \begin{pmatrix} 1 & 0 \\ ia & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=0}$$

$T_s = 2\pi/\omega_\beta$ is the synchrotron period, and the constant a is given by

Solution to the equations of motion

$$a = \frac{A}{2\omega_\beta \omega_s}(e^{\frac{i\omega_\beta T_s}{2}} - 1)$$

After half a synchrotron period, the roles of the particles are reversed, so that particle 2 sees no wake field, but particle 1 experiences the wake field from particle 2

By symmetry, we can write down, for the transverse co-ordinates of the particles after a full synchrotron period:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=T_s} \approx e^{-\frac{i\omega_\beta T_s}{2}} \begin{pmatrix} 1 & ia \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=T_s/2}$$

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=T_s} &\approx e^{-i\omega_\beta T_s} \begin{pmatrix} 1 & ia \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ia & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=0} \\ &\approx e^{-i\omega_\beta T_s} \begin{pmatrix} 1 - a^2 & ia \\ ia & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=0} \end{aligned}$$

Solution to the equations of motion

This is $(1-a^2-\lambda)(1-\lambda)+a^2=0$

Since the matrix has a unit determinant, the diagonalized matrix will also, so we have $\lambda_1\lambda_2=1$, $\lambda_{1,2}=\exp(\mp i\phi)$

and

$$(1-a^2-\exp(i\phi))(1-\exp(i\phi))+a^2=0$$

$$-\exp(i\phi)-(1-a^2)(\exp(i\phi))+\exp(i2\phi)=0$$

$$(1-a^2)=\exp(-i\phi)-1+\exp(i\phi)=2cos\phi-1$$

$$a^2=2-2cos\phi \qquad \Rightarrow a=2\sin\frac{\phi}{2} \quad , \quad a \le 2$$

The normal mode eigenvectors in the (y_1, y_2), basis are

$$\vec{\zeta_1}=\begin{pmatrix}-\exp(-i\frac{\phi}{2})\\ 1\end{pmatrix} \quad , \vec{\zeta_2}=\begin{pmatrix}\exp(i\frac{\phi}{2})\\ 1\end{pmatrix}$$

Transverse motion with wake fields: stability condition

"Transfer matrix" R for the motion of the two macroparticles over a full synchrotron period is

$$R = e^{-i\omega_\beta T_s}\begin{pmatrix} 1-a^2 & ia \\ ia & 1 \end{pmatrix}$$

The motion of the particles will be stable if the trace of the transfer matrix has magnitude less than 2, $|2-a^2| < 2$

Using $a = \frac{A}{2\omega_\beta\omega_s}(e^{\frac{i\omega_\beta T_s}{2}} - 1)$ and assuming the maximum magnitude for

a =|A|/$\omega_\beta\omega_s$, the stability condition becomes

$$\frac{c^2 e^2 N_0 W_o}{2\omega_\beta\omega_s C_0 E_0} < 2\,,$$

$$N_0 < \frac{16\pi^2 \nu_\beta \nu_s E_0}{e^2 W_0 C_0}$$

ν_β and ν_s are the betatron and synchrotron tunes.

Transverse motion with wake fields: stability condition

- It expresses a limit on the bunch population in a storage ring, above which the betatron oscillation amplitudes of particles in the bunch will increase exponentially, driven by the transverse wake fields. It is known as the "fast head-tail instability.

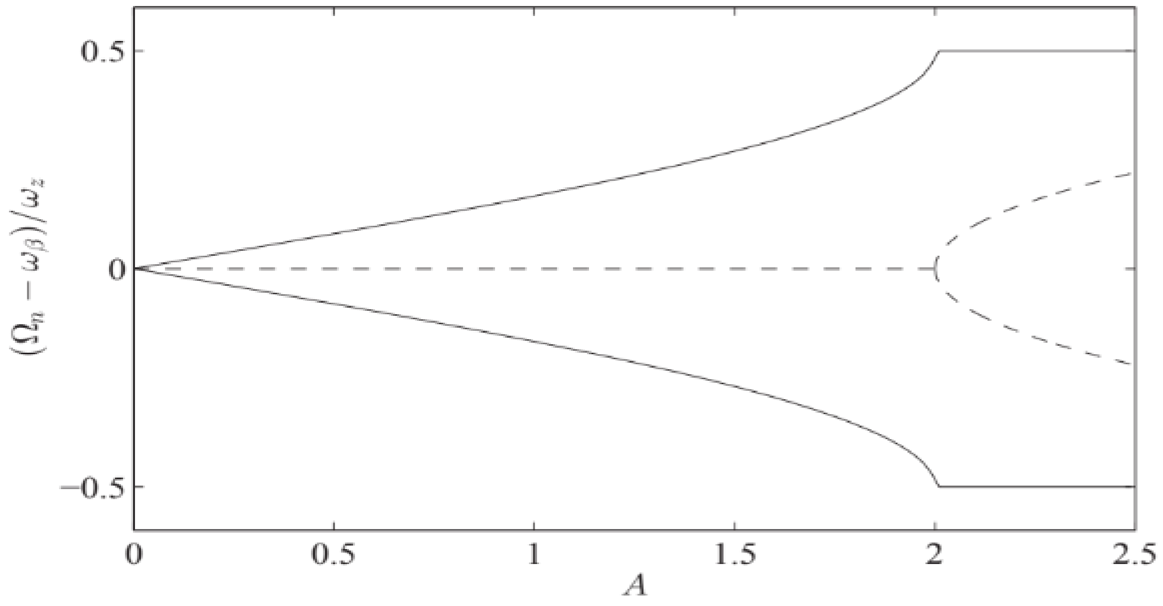
- The presence of transverse wake fields will also lead to a shift in the frequency of betatron oscillations performed by the particles.

- We can find the frequency shift from the solution to the equations of motion. The frequencies of the normal modes are given by the eigenvalues of the full 4×4 transfer matrix, describing the changes in co-ordinates and the transverse momenta of the particles over one synchrotron period.

Transverse motion with wake fields: stability condition

- If the wake fields are weak, then the normal mode frequencies are purely real: the particles perform betatron oscillations with constant amplitude.
- If the wake fields are increased (e.g. by increasing the bunch charge) the betatron frequencies acquire non-zero imaginary parts.
- At this point the oscillation amplitude of one mode will be damped, but the amplitude of the other mode will grow(exponentially)..

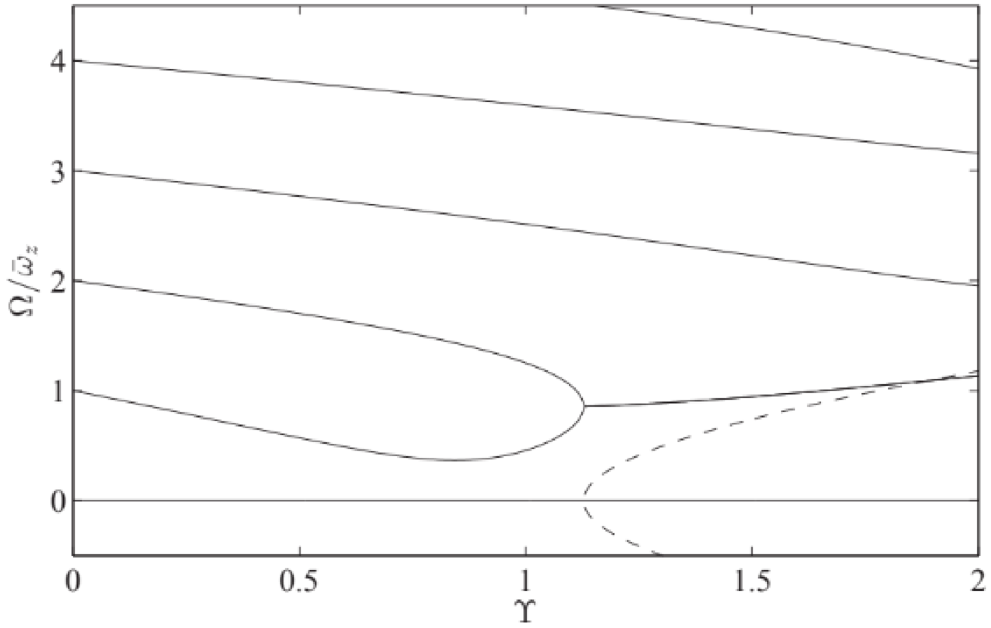
Transverse motion with wake fields: stability condition



Real and imaginary parts (solid and dashed lines, respectively) of normal mode frequency shifts (in units of the synchrotron frequency), as a function of wake field strength.

Transverse mode-coupling instability

- Wake fields shift the mode oscillation frequencies: if the wakefields are strong enough, the frequencies of two different mode scan become equal. When this happens, the frequencies acquire non-zero imaginary parts, indicating the exponential growth of the perturbation, i.e. the onset of an instability, known as the “transverse mode-coupling instability” (or strong head-tail instability)



Strong Head-tail instability : summary

- A simple model of a transverse single-bunch instability can be developed using a model of the bunch as two macroparticles.

- With some assumptions, and ignoring chromaticity, it is found that there is a threshold (in terms of the bunch charge, or the wake field strength) above which the betatron oscillations of the particles grow exponentially. The instability in this case is known as the fast head-tail instability.

- A more sophisticated model of transverse single-bunch instabilities, including the transverse mode-coupling instability can be developed using the Vlasov equation to describe the evolution of a charge distribution in transverse and longitudinal directions.

Effect of chromaticity

We have also ignored some important effects, including (for example) chromaticity. If chromaticity is taken into account, then the behaviour of the system is modified: the instability in this case is known as the "head-tail instability" (rather than the "fast head-tail instability").

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Head Tail Instability

Head-Tail Instability

- Betatron and synchrotron motions are decoupled in strong head-tail instability. Betatron frequency depends on energy error δ=ΔE/E of the particle. Betatron frequency for off-momentum particle is

$$\omega_\beta(\delta) = \omega_\beta(1+\xi\delta)$$

(ω_β: betatron frequency for on-momentum, ξ : chromaticity)

Accumulated betatron phase is

$$\phi_\beta(s) = \int \omega_\beta(\delta)\frac{ds}{c} = \int \omega_\beta(1+\xi\delta)\frac{ds}{c} = \omega_\beta\left(\frac{s}{c} + \xi\int\delta\frac{ds}{c}\right)$$

$$= \omega_\beta\left[\frac{s}{c} - \frac{\xi}{c\eta}\int z'\,ds\right]$$

$$= \omega_\beta\left[\frac{s}{c} - \frac{\xi}{c\eta}z(s)\right]$$

($z' = -\eta\delta$)
η: slippage factor
(η =1/γ^2-α, η=Δτ/τ/dp/p)

In the absence of wake field, deviation of betatron phase is determined by longitudinal position z, not δ.

Head-Tail Instability

We consider two macroparticles whose synchrotron oscillations

$$z_1 = \hat{z}\sin\left(\frac{\omega_s}{c}s\right), \qquad z_2 = -z_1$$

Particle 1 leads particle 2 during $0 < \frac{s}{c} < \frac{\pi}{\omega_s}$. and trails it during $\frac{\pi}{\omega_s} < \frac{s}{c} < \frac{2\pi}{\omega_s}$

Free betatron oscillations of two particles are

$$y_1(s) = \widetilde{y_1}e^{-i\phi_{\beta 1}(s)} = \widetilde{y_1}e^{-i[\omega_\beta \frac{s}{c} - \frac{\omega_\beta}{c\eta}\xi z_1]} = \widetilde{y_1}e^{-i[\omega_\beta \frac{s}{c} - \frac{\omega_\beta}{c\eta}\xi\hat{z}\sin\left(\frac{\omega_s}{c}s\right)]}$$

$$y_2(s) = \widetilde{y_2}e^{-i\phi_{\beta 2}(s)} = \widetilde{y_2}e^{-i[\omega_\beta \frac{s}{c} - \frac{\omega_\beta}{c\eta}\xi z_2]} = \widetilde{y_2}e^{-i[\omega_\beta \frac{s}{c} + \frac{\omega_\beta}{c\eta}\xi\hat{z}\sin\left(\frac{\omega_s}{c}s\right)]}$$

Leading particle lags in betatron phase relative to trailing particle if $\frac{\xi}{\eta} > 0$ The situation reverses if $\frac{\xi}{\eta} < 0$

$$\Delta\phi_\beta = \phi_{\beta 1} - \phi_{\beta 2} = \left(\omega_\beta \frac{s}{c} - \frac{\omega_\beta}{c\eta}\xi\hat{z}\sin\left(\frac{\omega_s}{c}s\right)\right) - \left(\omega_\beta \frac{s}{c} + \frac{\omega_\beta}{c\eta}\xi\hat{z}\sin\left(\frac{\omega_s}{c}s\right)\right) = -2\frac{\omega_\beta}{c\eta}\xi\hat{z}\sin\left(\frac{\omega_s}{c}s\right)$$

$\boxed{\frac{\xi\omega_\beta}{c\eta}\hat{z}}$ is head-tail phase. It is physical origin of head-tail instability.

Head-Tail Instability

The motion of particle 2 during $0 < \frac{s}{c} < \frac{\pi}{\omega_s}$ in the presence of wake field.

$$W_1(z) = \begin{cases} -W_0 & \text{if } 0 > z > -(\text{bunch length}) \\ 0 & \text{otherwise} \end{cases}$$

Eq. of motion $$y_2'' + \left[\frac{\omega_\beta(\delta_2)}{c}\right]^2 y_2 = \frac{N r_0 W_0}{2\gamma C} y_1$$

$$\omega_\beta(\delta_2) = \omega_\beta(1 + \xi\delta_2) = \omega_\beta\left[1 + \xi\left(-\frac{z'}{\eta}\right)\right] = \omega_\beta\left(1 + \frac{\xi \hat{z} \omega_s}{c\eta} \cos\frac{\omega_s s}{c}\right)$$

Head-Tail Instability

$$y_2'' = \frac{d}{ds}\left(\frac{d}{ds}y_2\right) = \frac{d}{ds}\left\{\frac{d}{ds}\left[\hat{y}_2 \exp\left(-i\omega_\beta \frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)\right]\right\}$$

$$= \frac{d}{ds}\left\{\frac{d\hat{y}_2}{ds}\exp\left(-i\omega_\beta \frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)\right.$$

$$\left. - i\left(\frac{\omega_\beta}{c} + \frac{\xi\omega_\beta}{c\eta}\hat{z}\frac{\omega_s}{c}\cos\frac{\omega_s s}{c}\right)\hat{y}_2 \exp\left(-i\omega_\beta \frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)\right\}$$

$\frac{\xi\omega_\beta}{c\eta}\hat{z}$ is very small.

$$\approx \frac{d^2\hat{y}_2}{ds^2}\exp\left(-i\omega_\beta \frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right) - 2i\frac{\omega_\beta}{c}\frac{d\hat{y}_2}{ds}\exp\left(-i\omega_\beta \frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)$$

$$-\frac{\omega_\beta^2}{c^2}\hat{y}_2 \exp\left(-i\omega_\beta \frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)$$

$\hat{y}_2$ is slowly varying $\Rightarrow$ $\hat{y}_2''$ term is ignored.

Head-Tail Instability

$$y_2'' \approx -2i\frac{\omega_\beta}{c}\frac{d\hat{y}_2}{ds}\exp\left(-i\omega_\beta\frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right) - \frac{\omega_\beta^2}{c^2}\hat{y}_2\exp\left(-i\omega_\beta\frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)$$

$$y_2'' + \frac{\omega_\beta^2}{c^2}\left(1 + \frac{\xi\hat{z}\omega_s}{c\eta}\cos\frac{\omega_s s}{c}\right)^2 y_2 = \frac{Nr_0W_0}{2\gamma C}y_1$$

$$y_2'' + \frac{\omega_\beta^2}{c^2}y_2 \approx \frac{Nr_0W_0}{2\gamma C}y_1$$

$$-2i\frac{\omega_\beta}{c}\frac{d\hat{y}_2}{ds}\exp\left(-i\omega_\beta\frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right) - \frac{\omega_\beta^2}{c^2}\hat{y}_2\exp\left(-i\omega_\beta\frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)$$
$$+\frac{\omega_\beta^2}{c^2}\hat{y}_2\exp\left(-i\omega_\beta\frac{s}{c} - i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right) \approx \frac{Nr_0W_0}{2\gamma C}\hat{y}_1\exp\left(-i\omega_\beta\frac{s}{c} + i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)$$

$$\therefore \hat{y}_2' \approx \frac{iNr_0W_0c}{4\gamma C\omega_\beta}\hat{y}_1(0)\exp\left(2i\frac{\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)$$

Head-Tail Instability

Head-tail phase $\frac{\xi\omega_\beta\hat{z}}{c\eta} \ll 1.$ Doing Taylor expansion

$$\hat{y}_2(s) = \hat{y}_2(0) + \frac{iNr_0W_0c}{4\gamma C\omega_\beta}\hat{y}_1(0)\int_0^s \left(1 + \frac{2i\xi\omega_\beta}{c\eta}\hat{z}\sin\frac{\omega_s s}{c}\right)ds$$

$$= \hat{y}_2(0) + \frac{iNr_0W_0c}{4\gamma C\omega_\beta}\hat{y}_1(0)\left[s + i\frac{2\xi\omega_\beta}{\eta\omega_s}\hat{z}\left(1 - \cos\frac{\omega_s s}{c}\right)\right]$$

① Resonant response term ② Chromatic term

① is responsible for the strong head-tail instability
② is small because it is proportional to the head-tail phase

Chromatic term is 90° out of phase from the resonant term.

Head-Tail Instability

$$\tilde{y}_2\Big|_{s=\frac{\pi c}{\omega_s}} = \tilde{y}_2(0) + \frac{iNr_0W_0c}{4\gamma C\omega_\beta}\tilde{y}_1(0)\left[\frac{\pi c}{\omega_s} + i\frac{2\xi\omega_\beta\hat{z}}{\eta\omega_s}\cdot 2\right]$$

$$= \tilde{y}_2\,(0) + \frac{i\pi Nr_0W_0c^2}{4\gamma C\omega_\beta\omega_s}\left(1 + i\frac{4\xi\omega_\beta\hat{z}}{\pi c\eta}\right)\tilde{y}_1(0)$$

$$= \tilde{y}_2\,(0) + i\Upsilon\,\tilde{y}_1(0)$$

$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = 0$$

$$\tilde{y}_1(s) = \tilde{y}_1\,(0)e^{-i\omega_\beta s/c} \qquad \omega_\beta \gg \omega_s$$

Transformation from $s = 0$ to $s = \frac{\pi c}{\omega_s}$

$$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=\frac{\pi c}{\omega_s}} = \begin{bmatrix}1 & 0\\ i\Upsilon & 1\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0}$$

$$\Upsilon = \frac{\pi Nr_0W_0c^2}{4\gamma C\omega_\beta\omega_s}\left(1 + i\frac{4\xi\omega_\beta\hat{z}}{\pi c\eta}\right)$$

$\xi \to 0$: *strong head − tail instavility*

$\xi \neq 0$: *head − tail instability*

Head-Tail Instability

$$for\ \frac{\pi c}{\omega_s} < s < \frac{2\pi c}{\omega_s}, \qquad \tilde{y}_1 \to \tilde{y}_2, \quad \tilde{y}_2 \to \tilde{y}_1$$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=\frac{2\pi c}{\omega_s}} = \begin{bmatrix} 1 & i\Upsilon \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{\frac{\pi c}{\omega_s}}$$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=\frac{2\pi c}{\omega_s}} = \begin{bmatrix} 1 & i\Upsilon \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i\Upsilon & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_0$$

$$= \begin{bmatrix} 1-\Upsilon^2 & i\Upsilon \\ i\Upsilon & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_0$$

Eigenvalue of this transformation matrix

$$(1-\Upsilon^2-\lambda)(1-\lambda)+\Upsilon^2 = 0$$

$$\lambda^2 - 2\lambda\left(1+\frac{\Upsilon^2}{2}\right)+1 = 0$$

$$\lambda^2 - 2\lambda\, cos\Upsilon + 1 = 0$$

$$\lambda - \left(e^{i\Upsilon}+e^{-i\Upsilon}\right)\lambda + e^{i\Upsilon}\cdot e^{-i\Upsilon} = 0$$

$$\lambda_{\pm} \approx e^{\pm i\Upsilon}$$

$+\ mode$: $two\ macroparticles\ oscillate\ in\ phase$

$-\ mode$: $two\ macroparticles\ oscillate\ out\ of\ phase$

Head-Tail Instability

- Imaginary part of Υ gives a growth of betatron oscillation.

$$\tau_{\pm}^{-1} = \frac{\pm i\Upsilon}{\mathrm{T}_s} = \mp \frac{N r_0 W_0 c \xi \hat{z}}{2\pi\gamma C \eta}$$

+ mode is damped if $\frac{\xi}{\eta} > 0$ and antidamped if $\frac{\xi}{\eta} < 0.$

- mode is damped if $\frac{\xi}{\eta} < 0$ and antidamped if $\frac{\xi}{\eta} > 0.$

- The only value of ξ that assures a stable beam is ξ=0.
  However, using the Vlasov equation, two particle model has overestimated growth rate of – mode.
- With presence of some stabilizing mechanisms (such as Landau damping, radiation damping), it leads us to choose slightly positive values for ξ for operation.