6. Longitudinal Dynamics

References

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2. A. Wolski, Beam Dynamics in Particle Accelerators, 2nd ed., World Scientific 2023
3. S. Y. Lee, Accelerator Physics, 4th ed., World Scientific 2021
4. H. Wiedemann, Particle Accelerator Physics, 4th ed., Springer Verlag 2015, 2019

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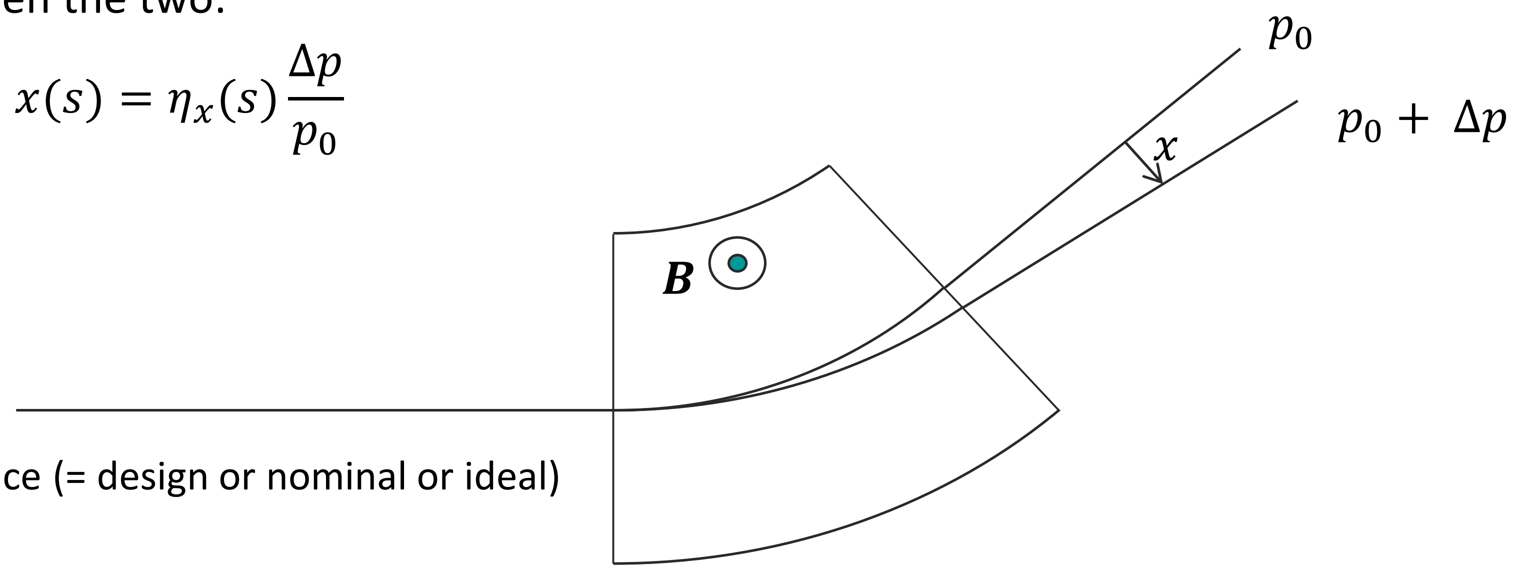
In Lecture 5, we have introduced the dispersion function. Before we discuss longitudinal dynamics in circular accelerators, let us introduce two more useful parameters that are related with the momentum of a particle; the momentum compaction factor and the phase slip (or frequency slippage) factor.

Recall

Dispersion function, η_x

Dispersion function relates the transverse orbit offset between the design (on-energy) particle and an off-energy particle divided by the relative difference in momentum between the two:

$$x(s) = \eta_x(s)\frac{\Delta p}{p_0}$$



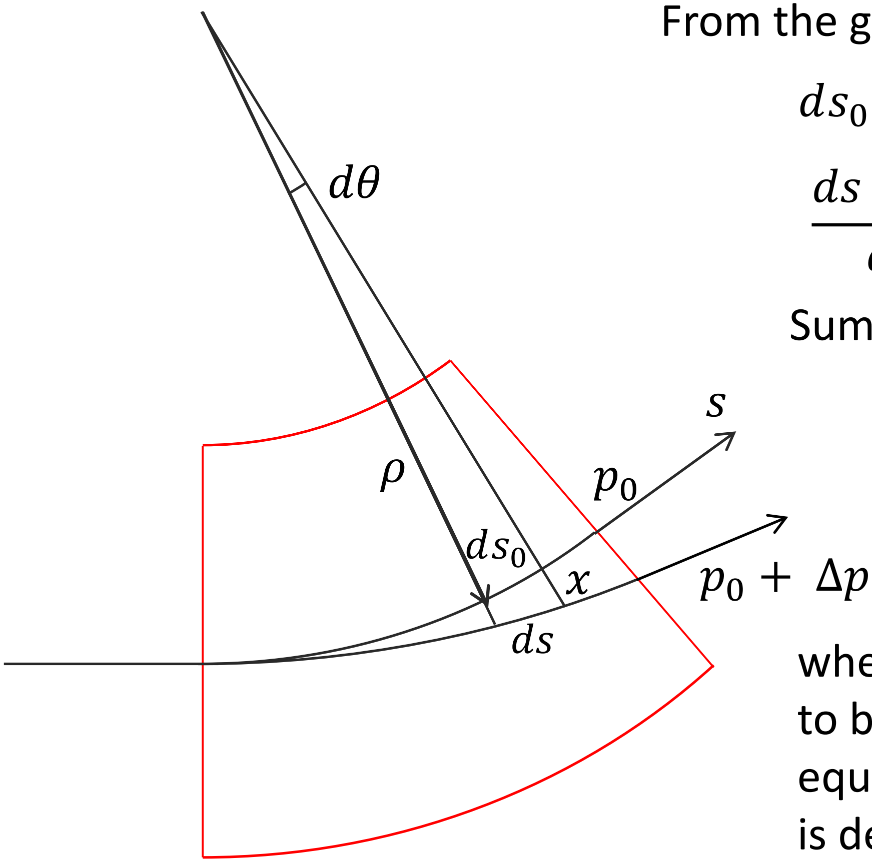
reference (= design or nominal or ideal) orbit

Thus the dispersion function is the momentum-dependent transverse orbit displacement.

Momentum compaction factor α_c

The momentum compaction factor is the momentum-dependent path length difference.

Consider the paths of two particles with different momenta, p_0 and $p_0 + \Delta p$ (Figure).



From the geometry we get the path elements

$$ds_0 = \rho d\theta, \qquad ds \approx (\rho + x)d\theta$$

$$\frac{ds - ds_0}{ds_0} \equiv \frac{dl}{ds_0} = \frac{x}{\rho} \quad \text{where} \quad dl = ds - ds_0$$

Summation around the ring

$$\int_0^{\Delta C} dl = \Delta C \equiv 2\pi \Delta R$$

R: effective radius of a ring

$$\Delta R = \frac{1}{2\pi}\int dl = \frac{1}{2\pi}\int_M \frac{x}{\rho} ds_0 \equiv \langle x \rangle_M$$

where the subscript M denotes the integration has to be evaluated in the magnets where $1/\rho$ is not equal to zero. The momentum compaction factor α_c is defined as, with $C_0 = 2\pi R_0$

$$\frac{\Delta C}{C_0} = \alpha_c \frac{\Delta p}{p_0}, \qquad \alpha_c = \frac{p_0}{R_0}\frac{\Delta R}{\Delta p} = \frac{p_0}{R_0}\frac{\langle x \rangle_M}{\Delta p} = \frac{\langle \eta_x \rangle_M}{R_0} = \frac{1}{2\pi R_0}\int_M \frac{\eta_x(s)}{\rho} ds \qquad (1)$$

So the path difference is due to the finite dispersion function.

e.g. PLS-2, α_c =0.0013

Phase slip factor η_p

It is useful to introduce the phase slip factor, which is the momentum-dependent (relative) time-of-flight difference:

$$\eta_p = \frac{\Delta T/T_0}{\Delta p/p_0} \qquad (2)$$

Time-of-flight of a particle with speed v to first order in small quantities is $T_0 = \frac{C_0}{v_0}$

$$T = \frac{C}{v} = \frac{C_0 + \Delta C}{v_0 + \Delta v} \approx (C_0 + \Delta C)\frac{v_0 - \Delta v}{v_0^2} \approx \frac{1}{v_0^2}(C_0 v_0 + v_0 \Delta C - C_0 \Delta v) = T_0 + \frac{v_0 \Delta C - C_0 \Delta v}{v_0^2}$$

The difference in time-of-flight is $\Delta T = T - T_0 = \frac{v_0 \Delta C - C_0 \Delta v}{v_0^2}$

or

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{T_0 v_0} - \frac{C_0 \Delta v}{T_0 v_0^2} = \underbrace{\frac{\Delta C}{C_0}}_{=\alpha_c \frac{\Delta p}{p_0}} - \underbrace{\frac{\Delta v}{v_0}}_{=\frac{1}{\gamma_0^2}\frac{\Delta p}{p_0}} = (\alpha_c - \frac{1}{\gamma_0^2})\frac{\Delta p}{p_0} \equiv \eta_p \frac{\Delta p}{p_0} \qquad (3)$$

Therefore

$$\eta_p = \frac{\Delta T/T_0}{\Delta p/p_0} = \alpha_c - \frac{1}{\gamma_0^2} \qquad (4)$$

η_p is called the phase slip factor (or frequency slippage factor or simply slippage factor), which as we shall see soon is an important parameter in longitudinal dynamics in synchrotrons. In electron synchrotron, $\eta_p \approx \alpha_c$. Note: η_p is not the dispersion function η.

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Synchrotron motion

Now we are ready to derive longitudinal equations of motion in synchrotron or storage ring. Let's consider a single particle moving through a synchrotron or storage ring. From the definition of our longitudinal canonical variable, $z = \frac{s}{\beta_0} - ct$, the change in z for a particle travelling with speed βc over one turn C_0 of the ring is

$$\Delta z = \frac{s + C_0}{\beta_0} - c(t+T) - \frac{s}{\beta_0} + ct = \frac{C_0}{\beta_0} - \frac{C}{\beta}, \qquad (C = \beta c T) \qquad (5)$$

Assuming that we can average the change in z over the entire circumference, we have

$$\frac{dz}{ds} \approx \frac{\Delta z}{C_0} = \frac{1}{\beta_0} - \frac{C}{C_0 \beta} = \frac{1}{\beta_0} - \frac{\beta c T}{c \beta_0 T_0 \beta} = \frac{1}{\beta_0}\left(1 - \frac{T}{T_0}\right) = -\frac{\eta_p}{\beta_0}\delta \qquad (6)$$

Using $\delta \approx p_t/\beta_0$, this can be expressed in terms of the canonical variable, p_t:

$$\frac{dz}{ds} = -\frac{\eta_p}{\beta_0^2} p_t \qquad (7)$$

This is one of the equations describing the longitudinal motion in a synchrotron or storage ring. To get the other (i.e. energy deviation) equation we consider the energy gained from the RF cavities and the energy lost by synchrotron radiation:

$$\Delta p_t = \frac{E - E_0}{p_0 c} = \frac{qV_{RF}}{p_0 c} \sin\left(\phi_{RF} - \frac{\omega_{RF}}{c} z\right) - \frac{U_0}{p_0 c} \qquad (8)$$

U_0: energy lost per turn by synchrotron radiation $\qquad$ ϕ_{RF} : fixed RF phase

Assuming again that we can take an average of Δp_t over the circumference, we have

$$\frac{dp_t}{ds} \approx \frac{\Delta p_t}{C_0} = \frac{qV_{RF}}{C_0 p_0 c} \sin\left(\phi_{RF} - \frac{\omega_{RF}}{c} z\right) - \frac{U_0}{C_0 p_0 c} \quad (9)$$

Taking the derivative of Eq. (7) with respect to s and substituting for $\frac{dp_t}{ds}$ from Eq. (9) we get

$$\frac{d^2 z}{ds^2} = -\frac{qV_{RF}}{C_0 p_0 c} \frac{\eta_p}{\beta_0^2} \sin\left(\phi_{RF} - \frac{\omega_{RF}}{c} z\right) + \frac{\eta_p}{\beta_0^2} \frac{U_0}{C_0 p_0 c} \quad (10)$$

If we set the RF phase such that $\phi_{RF} = \phi_s$, which is called the synchronous phase:

$$\sin \phi_s = \frac{U_0}{qV_{RF}} \quad (11)$$

Eqs. (9) and (10) then respectively become

$$\frac{dp_t}{ds} = -\frac{qV_{RF}}{C_0 p_0 c}\left[\sin \phi_s - \sin\left(\phi_s - \frac{\omega_{RF}}{c} z\right)\right] \quad (12)$$

$$\frac{d^2 z}{ds^2} = -\frac{qV_{RF}}{C_0 p_0 c} \frac{\eta_p}{\beta_0^2} \sin\left(\phi_s - \frac{\omega_{RF}}{c} z\right) + \frac{\eta_p}{\beta_0^2} \frac{qV_{RF}}{C_0 p_0 c} \sin \phi_s \quad (13)$$

Eq. (13) is a nonlinear equation in z so we linearize it to examine linear dynamics:

$$\sin\left(\phi_s - \frac{\omega_{RF}}{c} z\right) \approx \sin \phi_s - \frac{\omega_{RF}}{c} z \cos \phi_s \quad (14)$$

Substituting this into Eq. (12), the linearized energy equation is

$$\frac{dp_t}{ds} = -\frac{qV_{RF}}{C_0 p_0 c} \frac{\omega_{RF}}{c} z \cos \phi_s \quad (15)$$

And Eq. (13) when linearized becomes

$$\frac{d^2 z}{ds^2} + k_s^2\, z = 0 \tag{16}$$

where

$$k_s^2 = -\frac{qV_{RF}}{p_0 c}\frac{\omega_{RF}}{cC_0}\frac{\eta_p}{\beta_0^2}\cos\phi_s \tag{17}$$

We see that if $k_s^2 > 0$ or

$$qV_{RF}\eta_p\cos\phi_s < 0 \tag{18}$$

then the motion is stable; oscillation with the angular frequency $\omega_s \approx c\beta_0 k_s$ about the reference particle. The longitudinal oscillation is also called the synchrotron oscillation. The reference particle always sees a fixed RF phase ϕ_s.

The synchrotron frequency is the longitudinal oscillation frequency which is defined as

$$f_s = \frac{\omega_s}{2\pi} = \frac{c\beta_0 k_s}{2\pi} = \frac{1}{2\pi}\sqrt{-\frac{qV_{RF}}{p_0}\frac{\omega_{RF}\eta_p}{C_0}\cos\phi_s} \tag{19}$$

The synchrotron tune is defined as the number of synchrotron oscillations per one revolution of the synchronous particle:

$$\nu_s = \frac{\omega_s}{\omega_0} = \frac{c\beta_0 k_s}{2\pi/T_0} = \frac{C_0 k_s}{2\pi} = \frac{1}{2\pi}\sqrt{-\frac{qV_{RF}}{cp_0}\frac{\omega_{RF}C_0}{c}\frac{\eta_p}{\beta_0^2}\cos\phi_s} \tag{20}$$

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Here ω_0 is the angular revolution frequency of a reference particle. The RF angular frequency ω_{RF} must satisfy

$$\omega_{RF} = h\omega_0 \tag{21}$$

where $h = \frac{\omega_{RF}}{\omega_0}$ is called the harmonic number (e.g. PLS-2, h =470), which must be a positive integer for reference particle always to receive the same energy when it passes through the RF cavity.

Using Eq. (21), Eq. (20) can also be expressed by

$$\nu_s = \sqrt{-\frac{qV_{RF}h\eta_p}{2\pi\beta_0^2 E_0}\cos\phi_s} \tag{22}$$

The synchrotron tune is usually small, i.e. $\nu_s \ll 1$, (e.g. PLS−2, $\nu_s = 0.00850$ @V_{RF}=3.2 MV), compared to the betatron tune such that the synchrotron oscillation is a slow process compared with the betatron oscillation , (e.g. PLS−2, $\nu_x = 15.28$, $\nu_y = 8.18$).

Let's introduce the transition gamma, γ_t

$$\eta_p = \alpha_c - \frac{1}{\gamma^2} \equiv \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}, \tag{23}$$

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}} \quad \text{Transition gamma} \tag{24}$$

(e.g. PLS−2, $\gamma_t = 29.023$),

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From Eq. (18), we see that the condition for stable oscillation is assuming $qV_{RF} > 0$

$$\eta_p \cos\phi_s = (\alpha_c - \frac{1}{\gamma^2}) \cos\phi_s \approx \alpha_c \cos\phi_s < 0$$

Then from the condition for positive energy gain we find

$$\frac{\pi}{2} < \phi_s < \pi \quad \text{if } \alpha_c > \frac{1}{\gamma^2} \quad \text{or } \eta_p > 0 \quad \text{or } \gamma > \gamma_t$$

$$0 < \phi_s < \frac{\pi}{2} \quad \text{if } \alpha_c < \frac{1}{\gamma^2} \quad \text{or } \eta_p < 0 \quad \text{or } \gamma < \gamma_t$$

Principle of phase stability
V. I. Veksler 1944
E. M. McMillan 1945

Thus it is important where the synchronous phase lies depending on the sign of η_p.

We can understand this qualitatively by looking at the figure in the next slide.

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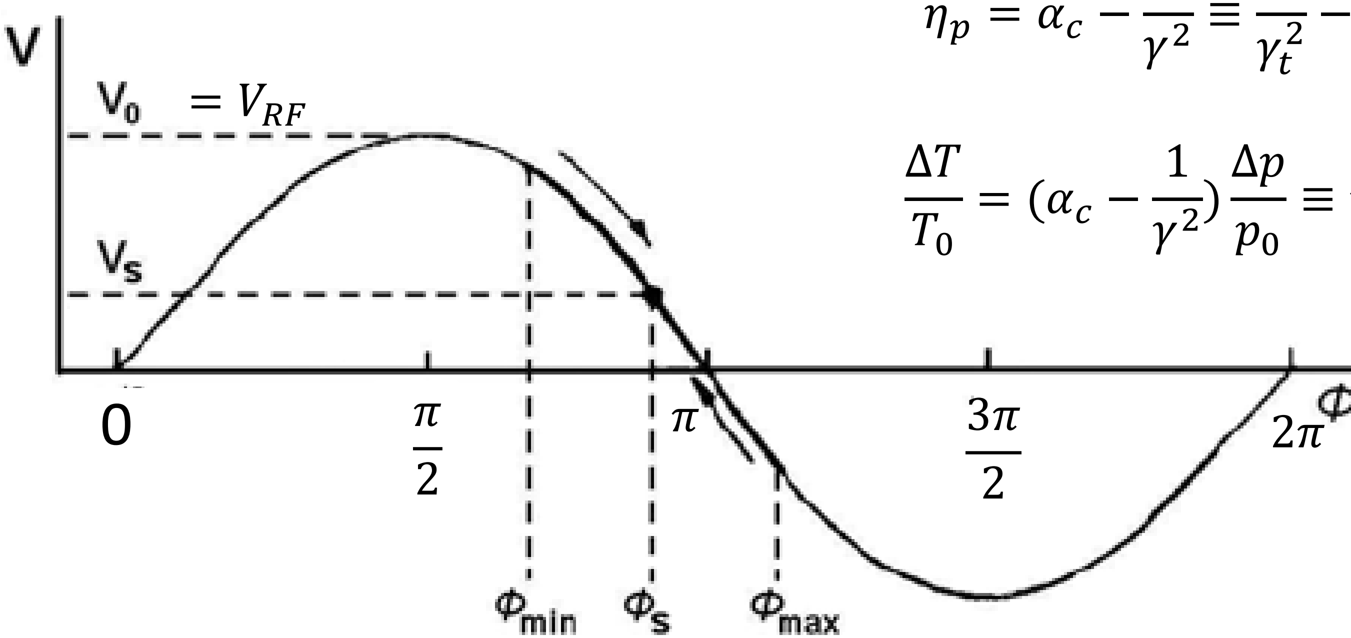
Principle of phase stability above transition

$$\eta_p = \alpha_c - \frac{1}{\gamma^2} \equiv \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} > 0$$

$$\frac{\Delta T}{T_0} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} \equiv \eta_p \frac{\Delta p}{p_0}$$



Above transition energy $\gamma > \gamma_t$, a particle with higher energy than the synchronous energy takes longer time to complete one revolution, i.e. $\Delta T > 0$. A particle which arrives the RF cavity earlier than the synchronous particle receives higher energy and therefore it will take longer time to complete one revolution around the ring; as a result it comes closer to the synchronous particle. A particle which arrives later than the synchronous particle, it will take shorter time and also it comes closer to the synchronous particle in the next turn.

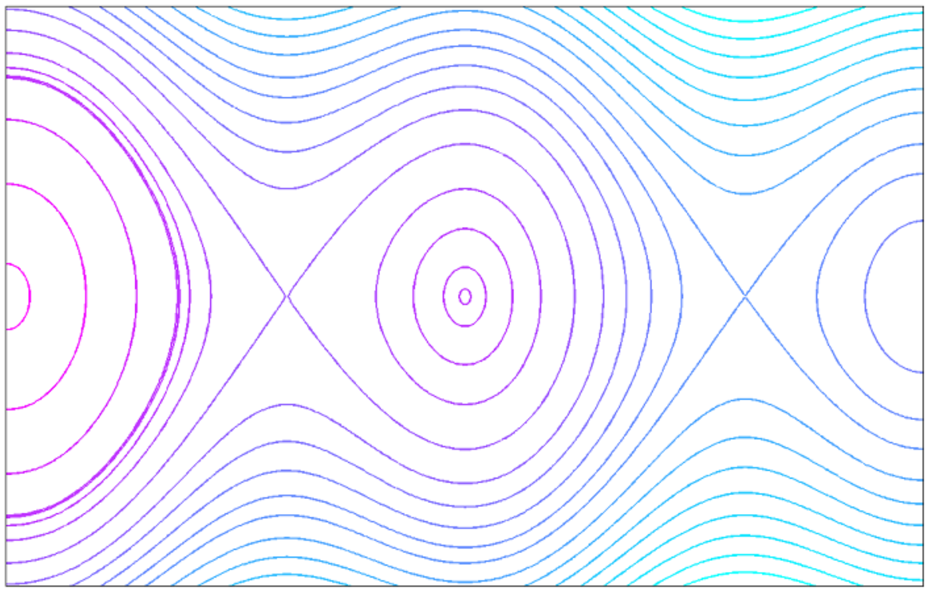
Hamiltonian for longitudinal motion

Eqs. (7) and (12) are the Hamilton's equations which can be derived from the Hamiltonian:

$$H = \frac{qV_{RF}}{\omega_{RF} C_0 p_0}\left[\frac{\omega_{RF}}{c} z \sin\phi_s - \cos\left(\phi_s - \frac{\omega_{RF}}{c} z\right)\right] - \frac{1}{2}\frac{\eta_p}{\beta_0^2} p_t^2 \qquad (25)$$

where Eq. (11) was used to eliminate U_0. Particles performing synchrotron oscillations will follow contours of constant value of the Hamiltonian in longitudinal phase space.

$$p_t = \frac{\Delta E}{c p_0}$$



$$z = \frac{C_0}{\beta_0} - \frac{C}{\beta}$$

Note that there are bounded regions where the contours form closed loops: particles in these regions perform stable oscillations. The curve forming the boundary of stable region is called the separatrix.

We have two first-order longitudinal equations of motion, Eqs. (7) and (12). Let's rewrite them introducing $\Delta\phi$, a change in RF phase:

$$\phi = \phi_s - \frac{\omega_{RF}}{c} z = \phi_s + \Delta\phi \qquad \Delta\phi = -\frac{\omega_{RF}}{c} z$$

and

$$\frac{dp_t}{ds} = \frac{qV_{RF}}{cC_0 p_0}(\sin\phi - \sin\phi_s) = \frac{qV_{RF}}{cC_0 p_0}[\sin(\phi_s + \Delta\phi) - \sin\phi_s] \qquad (26)$$

$$\frac{d\Delta\phi}{ds} = \frac{\omega_{RF}}{c}\frac{\eta_p}{\beta_0^2} p_t \qquad (27)$$

Combining these two, we get a second-order diff. equation as before, which can be written in the form

$$\frac{d^2\Delta\phi}{dt^2} + \frac{\omega_s^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0 \qquad (28)$$

where we have changed the independent variable from s to time t, $s = c\beta_0 t$ and used Eq. (19).

Equation (28) is a nonlinear pendulum equation. Although it is nonlinear, we can find in this case exact solutions given by Jacobi elliptic functions. But it is not illuminating. So we will not consider exact solutions here. Instead we discuss it based on integrals of motion. Multiplying by $d\Delta\phi/dt$ on both sides of Eq. (28), we get

$$\Delta\ddot{\phi}\Delta\dot{\phi} + \frac{\omega_s^2}{\cos\phi_s}\Delta\dot{\phi}(\sin\phi - \sin\phi_s) = 0 \quad \text{or} \quad \frac{d}{dt}\frac{\Delta\dot{\phi}^2}{2} + \frac{\omega_s^2}{\cos\phi_s}\Delta\dot{\phi}(\sin\phi - \sin\phi_s) = 0$$

$$\frac{d}{dt}\frac{\Delta\dot{\phi}^2}{2} - \frac{\omega_s^2}{\cos\phi_s}\frac{d}{dt}(\cos\phi + \Delta\phi\sin\phi_s) = 0 \qquad (29)$$

Integration yields

$$\frac{\Delta\dot{\phi}^2}{2} - \frac{\omega_s^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s) = \text{const.} \qquad 1^{st}\text{ integral} \qquad (30)$$

where we have changed from $\Delta\phi$ to ϕ $\quad \because \Delta\dot{\phi} = \dot{\phi}$

If the motion $\phi(t)$ is oscillatory and stable, there are two turning points of the motion in the phase space, ϕ_1 and ϕ_2, located on either side of ϕ_s for which the derivatives $\frac{d\phi_1}{dt} = 0$ and $\frac{d\phi_2}{dt} = 0$. To find these turning points, let's consider Eq. (30)

$$\frac{\Delta\dot{\phi}^2}{2} - \frac{\omega_s^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s) \rightarrow -\frac{\omega_s^2}{\cos\phi_s}(\cos\phi_1 + \phi_1\sin\phi_s)$$

$$= -\frac{\omega_s^2}{\cos\phi_s}(\cos\phi_2 + \phi_2\sin\phi_s) \qquad (31)$$

One of the phase extrema is at $\phi_1 = \pi - \phi_s$ because at $\phi_1 = \pi - \phi_s$ $\frac{d^2\phi}{dt^2} = 0$ [from Eq. (28)] so $d\phi/dt$ changes sign. This is called the "unstable fixed point".

Also $\frac{d^2\phi}{dt^2} = 0$ when $\phi = \phi_s$. This is called the "stable fixed point".

$$\underbrace{\frac{\Delta\dot{\phi}^2}{2}}_{\text{Kinetic energy}} \underbrace{- \frac{\omega_s^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s)}_{\text{Potential energy}} = -\frac{\omega_s^2}{\cos\phi_s}[\cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s) \qquad (32)$$

This is the equation of separatrix

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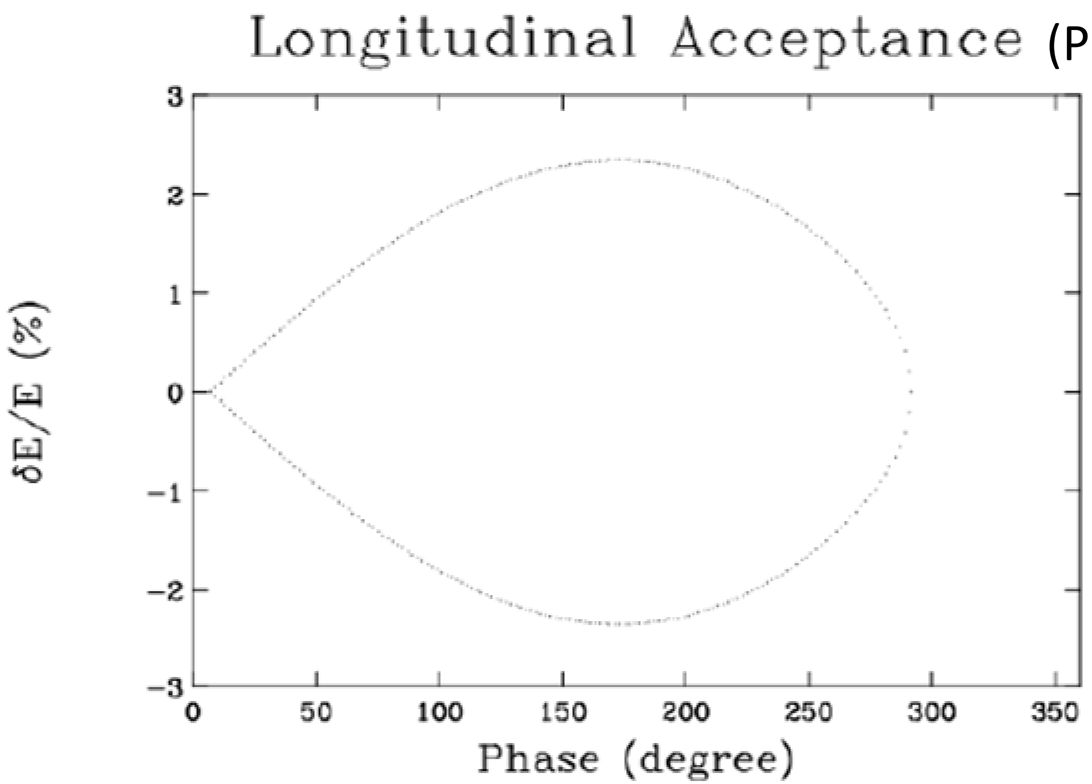
The other phase extremum can be found from Eq. (31) by solving

$$(\cos\phi_2 + \phi_2 \sin\phi_s) = [\cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s] \quad (33)$$

This can be solved numerically.



$h = 468$

$V_0 = 1.8\,\mathrm{MV}$

$f_{rf} = 500.0816\,\mathrm{MHz}$

$\phi_0 = 172.8298^\circ$

$\nu_s = 0.010969$

$f_s = 11.721\,\mathrm{kHz}$

$\left(\frac{\Delta E}{E_0}\right)_{rf} = \pm 2.34\%$

$\alpha = 0.00181$

$\phi_1 = 7.17^\circ,\ \phi_1 = 291.83^\circ$

RF bucket height (Energy acceptance)

The time (or s) derivative of the RF phase (or the energy change) reaches maximum (the second derivative is zero) at the synchronous phase, $\phi = \phi_s$. See Eq. (28).

The equation of the separatrix at this point becomes [from Eq. (32)]

$$\Delta\dot{\phi}^2_{max} = 2\omega_s^2[2 + (2\phi_s - \pi)\tan\phi_s]$$

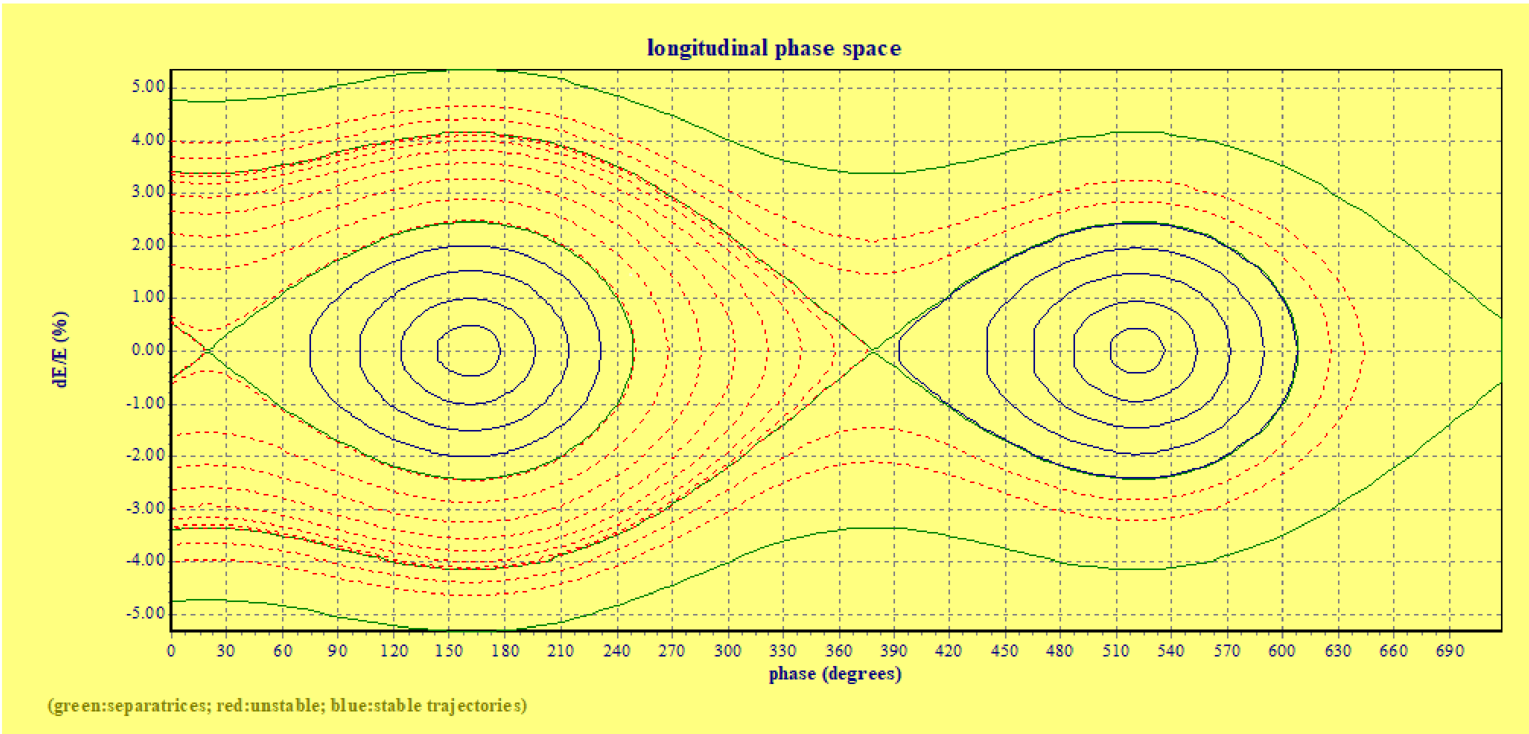
Replacing the time derivative of the phase from the first energy-phase relation, we get the RF bucket height which is the maximum energy acceptance of the synchrotron:

$$\left(\frac{\Delta E}{E_0}\right)_{max} = \pm\beta_0\left[\frac{qV_{RF}}{\pi h \eta_p E_0}\{(\pi - 2\phi_s)\sin\phi_s - 2\cos\phi_s\}\right]^{1/2} \qquad (34)$$

RF bucket height defines the energy acceptance which depends strongly on the choice of the synchronous phase. It plays an important role on injection matching and influences strongly on the beam life time in electron storage rings. In PLS-2, $\left(\frac{\Delta E}{E_0}\right)_{max}$ ~2.44% when $V_{RF} = 3.2$ MV.

A region within a separatrix, where the particles perform stable oscillations, is known as an RF bucket.

PLS-2, $C_0 = 281.82\ m$



$E = 3.0\ GeV, V_{RF} = 3.2\ MV, f_{RF} = 499.97323\ MHz, h = 470, U_0 = 1.04\ MeV$

$\phi_s = 160.994^{\circ}, f_s = 10.589\ kHz, \nu_s = 0.009954, \alpha_c = 0.001299, \sigma_l = 6.336\ mm$

$\frac{eV_{RF}}{U_0} = 3.071$ Over-voltage factor $\left(\frac{\Delta E}{E_0}\right)_{max} \sim 2.44\%$