5. Dispersion Function

References:
1. A. Wolski, Linear Dynamics, Lecture notes, November 2012
2. A. Wolski, Beam Dynamics in Particle Accelerators, 2nd ed., World Scientific

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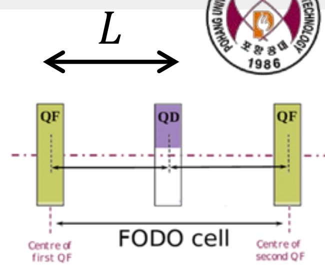
The transfer matrix for one cell of a FODO structure (without bends) is



$$R = \begin{pmatrix} 1-\frac{L^2}{2f^2} & \frac{L}{f}(L+2f) & 0 & 0 & 0 & 0 \\ \frac{L}{4f^3}(L-2f) & 1-\frac{L^2}{2f^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{L^2}{2f^2} & -\frac{L}{f}(L-2f) & 0 & 0 \\ 0 & 0 & -\frac{L}{4f^3}(L+2f) & 1-\frac{L^2}{2f^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

(HW 1: Prob. 4)

If we choose the drift length L and the quadrupole focal length f properly, particles oscillate in the transverse planes as they travel along the beamline. However, there are no longitudinal oscillations. From the above transfer matrix we can derive the longitudinal equations of motion:

$$\frac{dz}{ds} = \frac{p_t}{\beta_0^2\gamma_0^2}, \qquad \frac{dp_t}{ds} = 0 \quad (2)$$

Let us consider now how these equations are affected if we introduce dipole magnets into the beamline.

Recall the transfer matrix for a uniform sector bending magnet (i.e. $k_1 = 0$) with the dipole field matched to the reference momentum [Eq. (39) in Lecture 2,3]

$$R = \begin{pmatrix} \cos\theta & \rho\sin\theta & 0 & 0 & 0 & \dfrac{\rho(1-\cos\theta)}{\beta_0} \\ -\dfrac{\sin\theta}{\rho} & \cos\theta & 0 & 0 & 0 & \dfrac{\sin\theta}{\beta_0} \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\dfrac{\sin\theta}{\beta_0} & -\dfrac{\rho(1-\cos\theta)}{\beta_0} & 0 & 0 & 1 & \dfrac{L}{\beta_0^2\gamma_0^2} - \dfrac{L-\rho\sin\theta}{\beta_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where L is the path length of the reference particle in the bending magnet, which is usually called the length of the dipole magnet (should not be confused with the physical length of the magnet) and $\theta = \frac{L}{\rho}$ is the bending angle.

Note the non-zero R_{16} and R_{26} terms in this transfer matrix; these terms give the change in the horizontal coordinate and momentum with respect to changes in the energy deviation (actually p_t). They describe the "dispersion" introduced by the bending magnet. We can generalize the idea of dispersion to a dispersion function.

Consider a periodic beamline consisting of drifts, normal quadrupoles, and dipoles bending in the horizontal plane. In general, the transfer matrix for one periodic cell takes the form

$$R = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & \bullet \\ \bullet & \bullet & 0 & 0 & 0 & \bullet \\ 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (3)$$

where • represents some non-zero value. The vertical motion is decoupled from the horizontal and the longitudinal motion, but the horizontal motion and the longitudinal motion are coupled to each other. However, the horizontal motion has no dependence on the longitudinal coordinate z; this is because, in the absence of RF cavities, the fields have no time dependence.

Since the horizontal motion is completely decoupled from the vertical motion and from the longitudinal coordinate z, the horizontal motion can be described in terms of a 3×3 matrix as follows

$$\begin{pmatrix} x \\ \pi_x/p_0 \\ p_t \end{pmatrix}_{s=s_0+C_0} = \begin{pmatrix} R_{11} & R_{12} & R_{16} \\ R_{21} & R_{22} & R_{26} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \pi_x/p_0 \\ p_t \end{pmatrix}_{s=s_0} \qquad (4)$$

where C_0 is the length of a cell, measured along the reference orbit.

Consider a particle moving through the lattice with some energy deviation $\Delta E = E - E_0$. There exists a trajectory that this off-energy particle can follow. We can show the existence of this trajectory by actually calculating what it is. The periodic condition is

$$\begin{pmatrix} \tilde{x} \\ \tilde{\pi}_x/p_0 \\ p_t \end{pmatrix}_{s=s_0+C_0} = \begin{pmatrix} \tilde{x} \\ \tilde{\pi}_x/p_0 \\ p_t \end{pmatrix}_{s=s_0} \qquad (5)$$

From Eqs. (4) and (5), we find

$$\begin{pmatrix} \tilde{x} \\ \tilde{\pi}_x/p_0 \end{pmatrix} = \begin{pmatrix} 1 - R_{22} & -R_{12} \\ -R_{21} & 1 - R_{22} \end{pmatrix}^{-1} \begin{pmatrix} R_{16} \\ R_{26} \end{pmatrix} p_t \qquad (6)$$

The dispersion function η_x (and its conjugate η_{p_x}) is defined as the change in the horizontal orbit offset (and scaled canonical momentum offset) per relative momentum deviation $\delta = \frac{p-p_0}{p_0} = \frac{\Delta p}{p_0}$. But Eq. (6) is given with p_t instead of δ so we must find the relationship between these two variables; let's recall

$$p_t = \frac{E}{cp_0} - \frac{1}{\beta_0} = \frac{E - E_0}{cp_0} = \frac{\Delta E}{cp_0}$$

From $E^2 = p^2c^2 + m^2c^4$ we find $\Delta E = \dfrac{c^2 p \Delta p}{E}$

Thus, to first order in small variables we get

$$p_t = \frac{\Delta E}{cp_0} = \frac{cp\Delta p}{p_0 E} = \beta \frac{\Delta p}{p_0} = \beta\delta \approx \beta_0 \delta \qquad (7)$$

Therefore from Eq. (6) we have

$$\eta_x = \frac{x}{\delta}$$

$$\eta_{p_x} = \frac{\pi_x/p_0}{\delta}$$

$$\begin{pmatrix} \eta_x \\ \eta_{p_x} \end{pmatrix} = \beta_0 \begin{pmatrix} 1-R_{22} & -R_{12} \\ -R_{21} & 1-R_{22} \end{pmatrix}^{-1} \begin{pmatrix} R_{16} \\ R_{26} \end{pmatrix} \qquad (8)$$

If the inverse of the matrix in Eq. (8) exists, then the dispersion function also exists.

Strictly speaking, Eq. (8) is only valid if there are no RF cavities in the ring, and the particle energy is constant.

Once the dispersion function η_x and η_{p_x} are found with Eq. (8), we can propagate them to find their values at any place in the ring. Evolution of a particle in phase space from $s = s_0$ to s can be expressed as

$$\begin{pmatrix} x \\ \pi_x/p_0 \\ p_t \end{pmatrix}_s = M(s|s_0) \begin{pmatrix} x \\ \pi_x/p_0 \\ p_t \end{pmatrix}_{s_0} = M \begin{pmatrix} x_\beta \\ \pi_{x_\beta}/p_0 \\ 0 \end{pmatrix}_{s_0} + M \begin{pmatrix} x_\eta \\ \pi_{x_\eta}/p_0 \\ p_t \end{pmatrix}_{s_0} \qquad (9)$$

where M is a 3×3 transfer matrix from $s = s_0$ to s and we let $x = x_\beta + x_\eta$, $\frac{\pi_x}{p_0} = \frac{\pi_{x_\beta}}{p_0} + \frac{\pi_{x_\eta}}{p_0}$.

Since $x = x_\beta + x_\eta = x_\beta + \eta_x\delta$, Eq. (9) becomes

$$\frac{\pi_{x_\eta}}{p_0} = \eta_{p_x}\delta$$

$$\begin{pmatrix} x_\beta + \eta_x\delta \\ \pi_{x_\beta}/p_0 + \eta_{p_x}\delta \\ 0 + \beta_0\delta \end{pmatrix}_s = M \begin{pmatrix} x_\beta \\ \pi_{x_\beta}/p_0 \\ 0 \end{pmatrix}_0 + M \begin{pmatrix} \eta_x\delta \\ \eta_{p_x}\delta \\ \beta_0\delta \end{pmatrix}_0 \qquad (10)$$

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From this we have

$$\begin{pmatrix} \eta_x\delta \\ \eta_{p_x}\delta \\ \beta_0\delta \end{pmatrix}_s = \begin{pmatrix} R_{11} & R_{12} & R_{16} \\ R_{21} & R_{22} & R_{26} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x\delta \\ \eta_{p_x}\delta \\ \beta_0\delta \end{pmatrix}_{s=s_0}$$

where we set

$$M(s|s_0) = \begin{pmatrix} R_{11} & R_{12} & R_{16} \\ R_{21} & R_{22} & R_{26} \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} \eta_x \\ \eta_{p_x} \\ \beta_0 \end{pmatrix}_s = \begin{pmatrix} R_{11} & R_{12} & R_{16} \\ R_{21} & R_{22} & R_{26} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_{p_x} \\ \beta_0 \end{pmatrix}_{s=s_0} \qquad (11)$$

$$\begin{pmatrix} x_\beta \\ \pi_{x_\beta}/p_0 \\ 0 \end{pmatrix}_s = M \begin{pmatrix} x_\beta \\ \pi_{x_\beta}/p_0 \\ 0 \end{pmatrix}_{s=s_0}$$

Eq. (11) is the one for propagation of η_x and η_{p_x} from $s = s_0$ to s. If the transfer matrix M is for a unit (or cell) of length L in a circular accelerator, then we get from

$$\eta_x(s = s_0 + L) = R_{11}\eta_x(s_0) + R_{12}\eta_{p_x}(s_0) + \beta_0 R_{16}$$

Choosing s_0 to be the symmetry point, i.e. $\eta_{p_x}(s_0) = 0$, the dispersion function at the symmetry point is

$$\eta_x = \frac{\beta_0 R_{16}(L)}{1 - R_{11}(L)} \qquad (12)$$

Equation (11) indicates that η_x and $\eta_{p_x}$are eigenvector components of the revolution matrix M with eigenvalue one.

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