3. Sextupole in a periodic FODO cell

Consider a periodic FODO cell of total length L in the horizontal plane, with linear focusing function $K(s)$ and a single thin sextupole of integrated strength k_2 located at $s = s_0$. Thus,

$$S(s) = k_2 \delta\big(s - s_0\big).$$

The full hamiltonian can be expressed by

$$H(x, p; s) = H_0(x, p; s) + H_1(x; s)$$

where

$$\begin{aligned} H_0(x, p; s) &= \frac{p^2}{2} + \frac{K(s)}{2} x^2, \\ H_1(x; s) &= \frac{k_2}{6} x^3 \delta\big(s - s_0\big). \end{aligned}$$

By using Floquet transform, the Hamiltonian can be simplified to

$$H_0 = \frac{P^2 + X^2}{2}$$

and that the sextupole term at $s = s_0$ is

$$H_1 = \delta\big(s - s_0\big)\varepsilon X^3, \quad \varepsilon = \frac{k_2 \beta(s_0)^{3/2}}{6}.$$

Problem:

Introduce action–angle variables (J, ϕ) by

$$X = \sqrt{2J} \cos\phi, \quad P = -\sqrt{2J} \sin\phi,$$

so that the unperturbed Hamiltonian reads $H_0 = J$. Express the full Hamiltonian as

$$H(J, \phi; s) = J + \delta\big(s - s_0\big)\varepsilon (2J)^{3/2} \cos^3\phi.$$