Homework #2

2025 Accelerator Summer School

Due Aug. 7 (Thu.), 9:30 AM, 2025

1. Suppose an electron emits a photon of energy u: this leads to an instantaneous change in the energy deviation δ. The values of z and δ following the photon emission are

$$z' = z \simeq \frac{\alpha_p c}{\omega_s}\delta_0 \cos(\theta), \quad \delta' = \delta - \Delta\delta = \delta_0 \sin(\theta) - \frac{u}{E_0}. \tag{1}$$

   (a) Show that the amplitude of the energy oscillation after the photon emission is given by

$$\delta_0'^2 = \delta_0^2 - 2\delta_0 \frac{u}{E_0} \sin(\theta) + \frac{u^2}{E_0^2}.$$

   (b) Averaging over all particles in the bunch, show that the change in the mean square energy deviation $\Delta\sigma_\delta^2 = \langle \delta'^2 \rangle - \langle \delta^2 \rangle$ is given by

$$\Delta\sigma_\delta^2 = \frac{\langle u^2 \rangle}{2E_0^2}.$$

2. For the FODO lattice, we approximate the ratio of the synchrotron integrals as follows:

$$\frac{I_5}{I_2} = \left(4 + \frac{\rho^2}{f^2}\right)^{-\frac{3}{2}} \left[8 - \frac{\rho^2}{2f^2}\theta^2 + O(\theta^4)\right]. \tag{2}$$

   Assuming that $\rho \gg 2f \gg L/2$, and using approximation $j_x \approx 1$, show that the natural emittance in FODO lattice is given by

$$\varepsilon_{0,\mathrm{FODO}} \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3.$$

3. **[Sextupole in a Periodic FODO Cell]** Consider a periodic FODO cell of total length L in the horizontal plane, with linear focusing function $K(s)$ and a single thin sextupole of integrated strength k_2 located at $s = s_0$. Thus,

$$S(s) = k_2 \delta\big(s - s_0\big).$$

   (a) **Hamiltonian Formulation.** Derive the full Hamiltonian

$$H(x, p; s) = H_0(x, p; s) + H_1(x; s)$$

   where

$$H_0(x, p; s) = \frac{p^2}{2} + \frac{K(s)}{2}x^2,$$
$$H_1(x; s) = \frac{k_2}{6}x^3 \delta\big(s - s_0\big).$$

(b) **Floquet (Normalized Coordinates) Transformation.** Apply the canonical change of variables

$$x = \sqrt{\beta(s)}X, \quad p = \frac{1}{\sqrt{\beta(s)}}\big(P - \tfrac{\beta'(s)}{2}X\big)$$

where $\beta(s)$ is the periodic Courant-Snyder beta-function. Show that in the new variables the linear Hamiltonian becomes

$$H_0 = \frac{P^2 + X^2}{2}$$

and that the sextupole term at $s = s_0$ is

$$H_1 = \delta\big(s - s_0\big)\varepsilon X^3, \quad \varepsilon = \frac{k_2\beta(s_0)^{3/2}}{6}.$$

Include each algebraic step and note the choice $\beta'(s_0) = 0$ at the sextupole.

(c) **Action–Angle Variables.** Introduce action–angle variables (J, ϕ) by

$$X = \sqrt{2J}\cos\phi, \quad P = -\sqrt{2J}\sin\phi,$$

so that the unperturbed Hamiltonian reads $H_0 = J$. Express the full Hamiltonian as

$$H(J, \phi; s) = J + \delta\big(s - s_0\big)\varepsilon(2J)^{3/2}\cos^3\phi.$$

4. [**Nonlinear Hamiltonian and Resonance Harmonics**] In a three-fold symmetric storage ring with thin sextupoles located at positions $s = s_i$ $(i = 1, 2, 3)$, the one-turn Hamiltonian in action–angle variables is given by

$$H_0(J) = J,$$

and each sextupole contributes an instantaneous perturbation

$$H_{1,i}(J, \phi; s) = \delta(s - s_i)\varepsilon_i(2J)^{3/2}\cos^3\big(\phi + \phi_i\big), \quad \varepsilon_i = \frac{k_2\beta(s_i)^{3/2}}{6}.$$

(a) **Effective Nonlinear Hamiltonian.** Show that integrating the perturbations over one turn yields the secular (angle-averaged) Hamiltonian

$$H_{\text{eff}}(J) = J + \sum_m G_m(J),$$

where the Fourier coefficients

$$G_m(J) = \sum_{i=1}^{3}\varepsilon_i(2J)^{3/2}\cos\big(m\phi_i\big)$$

correspond to the m-th harmonic of the combined sextupole kicks.
Hint: The m-th harmonic coefficient can be extracted by the angle-average

$$G_m(J) = \frac{1}{2\pi}\int_0^{2\pi}\Big[\sum_{i=1}^{3}H_{1,i}(J, \phi; s)\Big]\cos(m\phi)d\phi,$$

which filters out all non-secular terms and retains only the slowly varying component.

(b) **Resonance Harmonics $m = 1, 3$.** Using the identity

$$\cos^3\theta = \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta,$$

demonstrate that only the $m = 1$ and $m = 3$ harmonics appear in each $H_{1,i}$. Discuss why these two harmonics are the primary drivers of horizontal nonlinear resonances in the three-fold symmetric lattice.

5. [**Transfer Map for a Sextupole Magnet**] Consider a relativistic particle moving through a thin sextupole magnet of length L with Hamiltonian

$$H(x, p_x) = -\sqrt{1 - p_x^2} + \frac{k_2}{6}x^3, \tag{3}$$

where x is the transverse coordinate, p_x the transverse momentum, and k_2 the sextupole strength.

(a) **Full Lie Transformation.** Apply the full Lie transformation to the phase-space variables:

$$(x, p_x) \mapsto e^{-L,:H:}\big(x, p_x\big). \tag{4}$$

Using the Poisson-bracket expansion

$$e^{-L:H:}f = f - L\big[H, f\big] + \frac{L^2}{2!}\Big[H, \big[H, f\big]\Big] + \mathcal{O}(L^3), \tag{5}$$

derive the transfer map to second order in L

(b) **Kick–Drift Splitting.** Split the Hamiltonian into drift and kick parts:

$$H_{\text{drift}}(p_x) = -\sqrt{1 - p_x^2}, \tag{6}$$

$$H_{\text{kick}}(x) = \frac{k_2}{6}x^3. \tag{7}$$

Use the symmetric composition

$$e^{-L:H:} \approx e^{-\frac{L}{2}:H_{\text{drift}}:}e^{-L:H_{\text{kick}}:}e^{-\frac{L}{2}:H_{\text{drift}}:} = \mathcal{M} + \mathcal{O}(L^3). \tag{8}$$

Expand each exponential to $\mathcal{O}(L^2)$ and show explicitly:

$$\mathcal{M}: \quad x_{\text{out}} = x + \big[\cdots\big]L + \big[\cdots\big]L^2, \tag{9}$$

$$p_{x,\text{out}} = p_x + \big[\cdots\big]L + \big[\cdots\big]L^2. \tag{10}$$

(c) **Symplecticity Check.** Compute the Jacobian matrix of each map:

$$M = \begin{pmatrix} \frac{\partial x_{\text{out}}}{\partial x} & \frac{\partial x_{\text{out}}}{\partial p_x} \\ \frac{\partial p_{x,\text{out}}}{\partial x} & \frac{\partial p_{x,\text{out}}}{\partial p_x} \end{pmatrix} = M^{(0)} + LM^{(1)} + L^2M^{(2)} + \mathcal{O}(L^3). \tag{11}$$

Verify to second order that

$$M^TJM = J + \mathcal{O}(L^3), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12}$$

Conclude both the full Lie map and the kick–drift map are symplectic to $\mathcal{O}(L^2)$.