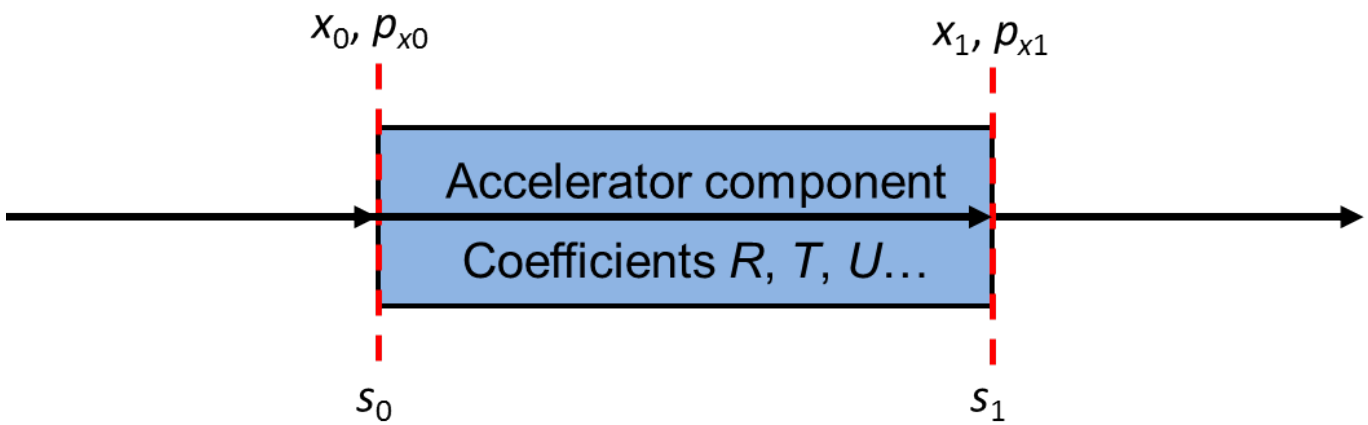
Nonlinear Beam Dynamics

Part 2: Hamiltonian Mechanics

Chong Shik Park, Ph.D

Korea University, Sejong

So far, our analysis has been based on transfer maps represented in the form of Taylor series.



For example:

$$x_1 = R_{11}x_0 + R_{12}p_{x0} + T_{111}x_0^2 + T_{112}x_0p_{x0} + T_{122}p_{x0}^2 + \cdots \quad (1)$$

$$p_{x1} = R_{21}x_0 + R_{22}p_{x0} + T_{211}x_0^2 + T_{212}x_0p_{x0} + T_{222}p_{x0}^2 + \cdots \quad (2)$$

In simple cases, Taylor maps can be very useful and convenient.

For example, once the coefficients in a Taylor map for a particular component (or section) in a beamline have been worked out, a Taylor map can be readily implemented in a tracking code.

Also, the coefficients in a Taylor map contain information about the dynamics of the system.

However, in several degrees of freedom Taylor series quickly become cumbersome, especially where high-order effects are important.

It is also difficult to enforce certain desirable properties of the dynamics (in particular, symplecticity) when using Taylor maps.

To make progress, we need to use more sophisticated tools.

Some very powerful methods for analysis of nonlinear systems are based on Hamiltonian mechanics. In this lecture, we review the basic principles of Hamiltonian mechanics in the context of accelerator beam dynamics.

In particular, we shall:

1. review Hamilton's equations;
2. discuss the significance of symplecticity;
3. derive (and solve) the nonlinear equations of motion for a drift space in an accelerator;
4. review canonical transformations, and introduce action–angle variables.

By the end of this lecture, you should be able to:

- derive the equations of motion for a dynamical system with a given Hamiltonian;
- be able to express relationships between different sets of variables in the form of canonical transformations.

In Hamiltonian mechanics, the state of a particle is specified by giving particular values for a set of dynamical variables.

The dynamical variables occur in pairs, with each pair consisting of a *co-ordinate* and a *conjugate momentum*.

The dynamics of the particle are described by expressing the dynamical variables as functions of an independent variable (for example, time t).

For example, for a particle of mass m performing simple harmonic motion with frequency ω, the equations of motion are derived from Newton's second law:

$$\frac{dq}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = -m\omega^2 q, \tag{3}$$

where $q = q(t)$ is the position of the particle at time t, and $p = p(t)$ is the momentum of the particle at time t.

The equations of motion have solution:

$$q(t) = a\cos(\omega t + \phi_0), \tag{4}$$

$$p(t) = -m\omega a \sin(\omega t + \phi_0), \tag{5}$$

where a (the amplitude) and ϕ_0 (a constant phase) are constants determined by the initial conditions.

In Newtonian mechanics, the equations of motion for a particle in a specific case are determined by the force F on the particle:

$$\frac{dq}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = F. \tag{6}$$

In Hamiltonian mechanics, the equations of motion are derived from a function called the *Hamiltonian*.

The Hamiltonian is a function of the *dynamical variables* and (in general) the *independent variable*.

Given the Hamiltonian, we can use Hamilton's equations to construct the equations of motion in a particular case.

If the dynamical variables are (q_i, p_i) and the independent variable is t, Hamilton's equations are:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{7}$$

where $H = H(q_i, p_i; t)$ is the Hamiltonian.

In Newtonian mechanics, the momentum of a particle is generally given by the product of the mass and the velocity, i.e. $p = m\,dq/dt$.

In Hamiltonian mechanics, an expression for the momentum can be obtained from Hamilton's equations: the momentum is not always the mass times the velocity.

The Hamiltonian must be expressed in terms of the co-ordinates and the conjugate momenta, and *not* in terms of the velocities.

In some simple cases, the Hamiltonian takes the form:

$$H = T + V, \tag{8}$$

where T is the kinetic energy of the particle, and V is the potential energy.

Consider a particle with mass m and co-ordinate q, moving in a potential:

$$V = \frac{1}{2}kq^2. \tag{9}$$

In this case (not in general), the momentum is $p = m\dot{q}$. Then, the Hamiltonian takes the form:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kq^2. \tag{10}$$

The first of Hamilton's equations (7) gives:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}. \tag{11}$$

This tells us that the momentum in this case corresponds to the usual mechanical momentum; i.e. the product of the mass and the velocity, $p = m\,dq/dt$.

The second of Hamilton's equations (7) gives:

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -kq. \tag{12}$$

This expresses Newton's second law of motion for a force $-kq$.

Combining the two equations gives the familiar second-order equation of motion for a simple harmonic oscillator:

$$\frac{d^2q}{dt^2} = -\omega^2 q, \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}. \tag{13}$$

To apply Hamiltonian mechanics to a given system, we need to:

- define the dynamical variables;
- define the independent variable;
- write down the Hamiltonian that defines the physics of the system.

In the case of a relativistic particle in an electromagnetic field, we can choose to work in a Cartesian co-ordinate system, with co-ordinates x, y, z, and conjugate momenta p_x, p_y, p_z.

We can choose the time t as the independent variable.

The Hamiltonian is:

$$H = \sqrt{c^2(\vec{p} - q\vec{A})^2 + m^2c^4} + q\varphi, \tag{14}$$

where:

- c is the speed of light in free space,
- m is the mass of the particle,
- q is the electric charge of the particle,
- $\vec{A}$ is the vector potential,
- and φ is the scalar potential.

Applying Hamilton's equations with the Hamiltonian (14), we find that the equations of motion for the co-ordinates give:

$$p_x = \gamma m \frac{dx}{dt} + qA_x, \tag{15}$$

and similarly for p_y and p_z.

The equations of motion for the momenta lead to:

$$\frac{d^2x}{dt^2} = \frac{q}{\gamma m}\left(E_x + \left[\vec{v} \times \vec{B}\right]_x\right), \tag{16}$$

and similarly for y and z.

$\vec{v} = (\dot{x}, \dot{y}, \dot{z})$ is the velocity of the particle.

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ are derived from the potentials $\vec{A}$ and φ in the usual way:

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}. \tag{17}$$

Equation (16) is the equation of motion that we would write down for a relativistic particle in an electromagnetic field, using Newton's second law and the Lorentz force equation.

There are formal methods to derive the Hamiltonian (starting from the Lagrangian for a given system).

Ultimately, the form of the Hamiltonian can be justified by whether it gives the correct (observed) dynamics.

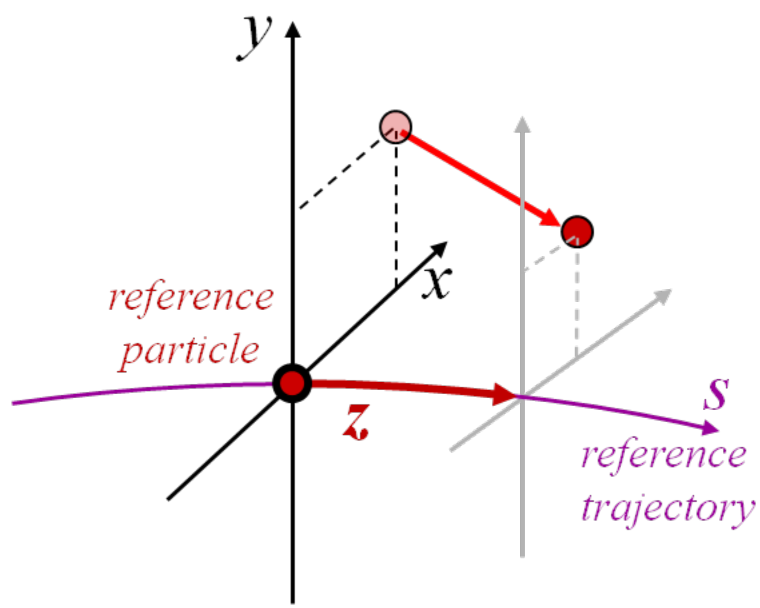
In a particle accelerator:

- It is convenient to work in a curved co-ordinate system, with x and y the transverse co-ordinates with respect to some "reference trajectory".

- It is convenient to choose the distance s along the reference trajectory as the independent variable.

For simplicity, we shall assume that the reference trajectory lies in a horizontal plane.

Then, the reference trajectory can be defined by a sequence of straight lines of given lengths, joined by lines with given lengths and curvatures.



The longitudinal co-ordinate z of a particle is defined:

$$z(s) = c(t_0 - t), \tag{18}$$

where the particle arrives at s at time t, and a reference particle (travelling along the reference trajectory with momentum P_0) arrives at s at time t_0.

If we were using Cartesian co-ordinates, and using time as the independent variable, then we could just use the Hamiltonian (14) for a particle in an accelerator beamline.

The dynamics at any point along the beamline would be defined by the scalar and vector potential at the given point.

Since we are using a curved co-ordinate system, with path length s as the independent variable, the Hamiltonian takes a rather more complicated form.

It is possible to derive the accelerator Hamiltonian starting from (14); but to save time we shall just quote the result.

For a full derivation of the accelerator Hamiltonian, see "Beam Dynamics in High Energy Particle Accelerators" by A. Wolski.

The Hamiltonian for a charged particle in an accelerator beamline is:

$$H = -(1+hx)\sqrt{\left(\frac{1}{\beta_0}+\delta-\frac{q\phi}{P_0 c}\right)^2 - \left(p_x - \frac{q}{P_0}A_x\right)^2 - \left(p_y - \frac{q}{P_0}A_y\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - (1+hx)\frac{q}{P_0}A_s + \frac{\delta}{\beta_0}. \tag{19}$$

Here, $P_0 = \beta_0\gamma_0 mc$ is the reference momentum (i.e. the momentum of the reference particle, with velocity $\beta_0 c$).

h is the curvature of the reference trajectory (assumed to lie in the $x-s$ plane):

$$h = \frac{1}{\rho}, \tag{20}$$

where ρ is the local radius of curvature.

The momenta conjugate to the co-ordinates x and y are given by:

$$p_x = \frac{\gamma m \dot{x} + qA_x}{P_0}, \qquad p_y = \frac{\gamma m \dot{y} + qA_y}{P_0}. \tag{21}$$

Here, $\dot{x}$ and $\dot{y}$ are the transverse components of the velocity (i.e. the time derivatives of the transverse co-ordinates).

γ is the relativistic factor for the particle (not necessarily equal to γ_0).

The longitudinal dynamical variables are (z, δ), where:

$$z = c(t_0 - t), \tag{22}$$

- t_0 is the time at which the reference particle arrives at a location s along the reference trajectory,
- t is the time at which the given particle crosses the plane perpendicular to the reference trajectory at s.

Note that if $t < t_0$, the chosen particle arrives at s sooner than the reference particle, i.e. the chosen particle is *ahead* of the reference particle.

The longitudinal conjugate momentum δ is defined by:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} = \frac{E - E_0}{\beta_0 E_0}, \tag{23}$$

where E is the kinetic energy of the particle, and E_0 is the kinetic energy of a particle with the reference momentum P_0.

As an example, let us consider the Hamiltonian in a drift space, where $h = 0$, and there are no electric or magnetic fields (so we can take the scalar and vector potentials to be zero):

$$H = -\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{\delta}{\beta_0}. \tag{24}$$

The Hamiltonian has no dependence on the co-ordinates x, y or δ. This means, from Hamilton's equations, that the momenta are conserved:

$$\frac{dp_x}{ds} = -\frac{\partial H}{\partial x} = 0, \tag{25}$$

$$\frac{dp_y}{ds} = -\frac{\partial H}{\partial y} = 0, \tag{26}$$

$$\frac{d\delta}{ds} = -\frac{\partial H}{\partial z} = 0. \tag{27}$$

The equations of motion for the co-ordinates are also reasonably straightforward:

$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x} = \frac{p_x}{p_s}, \tag{28}$$

$$\frac{dy}{ds} = \frac{\partial H}{\partial p_y} = \frac{p_y}{p_s}, \tag{29}$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta}{p_s}, \tag{30}$$

where we have defined p_s (not a *dynamical variable*!) as:

$$p_s = \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}}. \tag{31}$$

Note that since p_x, p_y and δ are constants of the motion, p_s is constant.

From the above results, it is possible to write the map for a drift space in closed form.

For the transverse variables, we have:

$$x_1 = x_0 + \frac{p_{x0}}{p_s}\Delta s, \qquad p_{x1} = p_{x0}, \tag{32}$$

$$y_1 = y_0 + \frac{p_{y0}}{p_s}\Delta s, \qquad p_{y1} = p_{y0}, \tag{33}$$

where $x_0 = x(s_0)$, $x_1 = x(s_0 + \Delta s)$, and similarly for the other variables.

And for the longitudinal variables, we have:

$$z_1 = z_0 + \left(\frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta_0}{p_s} \right) \Delta s, \quad \delta_1 = \delta_0. \tag{34}$$

The map has a nonlinear dependence on the momenta p_x, p_y and δ. However, the nonlinear effects only become significant when the values of the momenta become very large.

To illustrate this, consider the case $p_{y0} = \delta_0 = 0$. Then:

$$p_s = \sqrt{1 - p_{x0}^2}. \tag{35}$$

In this case:

$$p_x = \frac{\gamma_0 m \dot{x}}{P_0}, \quad \text{so that} \quad \lim_{\dot{x} \to \beta_0 c} p_x = 1. \tag{36}$$

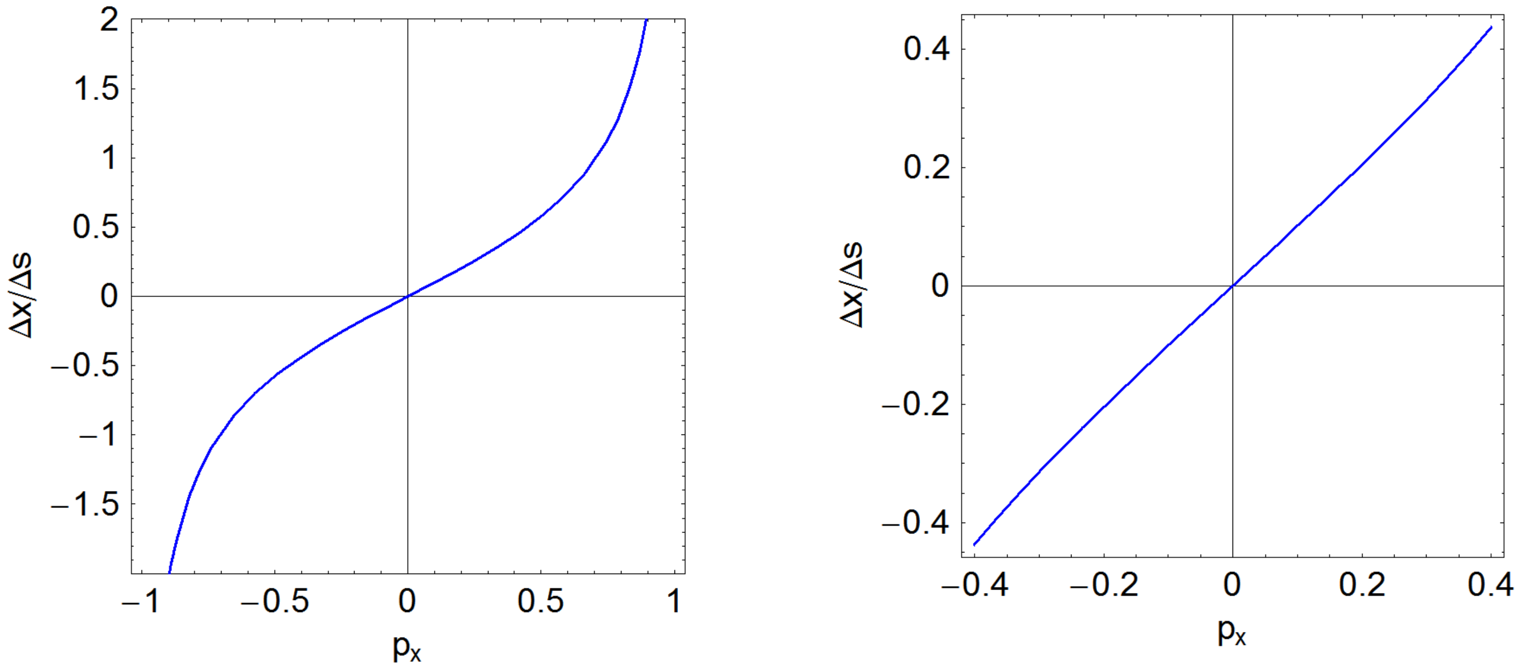
The horizontal momentum p_x has a maximum value of 1, which occurs when the particle is travelling perpendicular to the reference trajectory.

Let us now plot:

$$\frac{\Delta x}{\Delta s} = \frac{p_{x0}}{\sqrt{1 - p_{x0}^2}}, \tag{37}$$

(where $\Delta x = x_1 - x_0$) as a function of p_{x0}.

There is a significant deviation from linearity when p_{x0} is larger than about 0.1.



In the case that $p_{x0} = p_{y0} = 0$, the particle is travelling parallel to the reference trajectory. Then, the Hamiltonian becomes:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}}. \tag{38}$$

It follows that the equation of motion for the longitudinal co-ordinate is:

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta}{\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}}}. \tag{39}$$

Since, from equation (23), we have:

$$\delta = \frac{E - E_0}{\beta_0 E_0} = \frac{\gamma - \gamma_0}{\beta_0\gamma_0}, \quad \therefore \quad \frac{1}{\beta_0} + \delta = \frac{\gamma}{\beta_0\gamma_0}, \tag{40}$$

we find that:

$$\frac{dz}{ds} = \frac{1}{\beta_0} - \frac{1}{\beta}, \tag{41}$$

which is consistent with our interpretation of z, equation (22).

Using a Hamiltonian approach, even the map for a drift space is rather complicated.

It is possible to describe the dynamics using different variables, that simplify the map. For example, instead of using p_x and p_y, we could define:

$$x' = \frac{dx}{ds}, \quad \text{and} \quad y' = \frac{dy}{ds}. \tag{42}$$

Then, the (transverse) map for a drift space would simply be:

$$x_1 = x_0 + x'_0\,\Delta s, \qquad x'_1 = x'_0, \tag{43}$$

$$y_1 = y_0 + y'_0\,\Delta s, \qquad y'_1 = y'_0, \tag{44}$$

with no dependence at all on the energy deviation.

This looks much simpler – why do we bother with the Hamiltonian? There are three reasons...

- Hamiltonian mechanics provides a highly systematic framework for constructing the equations of motion for a relativistic particle in even quite complicated electromagnetic fields.

- Hamiltonian mechanics provides the basis for some powerful analytical techniques for modelling and analysis of beam dynamics.

- Proper use of Hamiltonian methods ensures the conservation of phase space volumes (Liouville's theorem), which is a property of the physics of particles in accelerators (nelgecting synchrotron radiation and collective effects).

Let $\vec{x}$ be a vector constructed from the phase space variables. If the values of the phase space variables at position $s + \Delta s$ on the reference trajectory are given by $\vec{X} = \vec{X}\left(\vec{x}(s); \Delta s\right)$, then:

$$J^{\mathsf{T}} S J = S, \tag{45}$$

where J is the Jacobian of the transformation from s to $s + \Delta s$:

$$J_{ij} = \frac{\partial X_i}{\partial x_j}, \tag{46}$$

and S is a block-diagonal matrix constructed from 2×2 antisymmetric matrices S_2:

$$S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{47}$$

Any matrix that satisfies equation (45) is said to be a *symplectic* matrix.

Since the determinant of S is unity:

$$|S| = 1, \tag{48}$$

it follows from (45) that if J is the Jacobian of a symplectic map, then:

$$|J|^2 = 1, \quad \text{i.e.} \quad |J| = \pm 1, \tag{49}$$

where $|J|$ is the determinant of J.

For a map to be symplectic, it is a necessary (but not sufficient) condition for the Jacobian to have determinant ± 1.

It follows immediately from this that symplectic maps preserve volumes in phase space:

$$\int \cdots \int d\vec{X} = \int \cdots \int |J| \, d\vec{x} = \pm \int \cdots \int d\vec{x}. \tag{50}$$

In the context of Hamilton mechanics, equation (50) is called Liouville's theorem.

In accelerator beam dynamics, Liouville's theorem says that as a bunch of particles is transported along a beamline (neglecting radiation and interactions between the particles) the volume of phase space occupied by the particles remains constant.

The total volume in phase space is one of a number of invariants of Hamiltonian systems, known as Poincaré invariants. The others are not so easily expressed as the volume of an element in phase space, and since we do not need them in this course, we do not discuss them further.

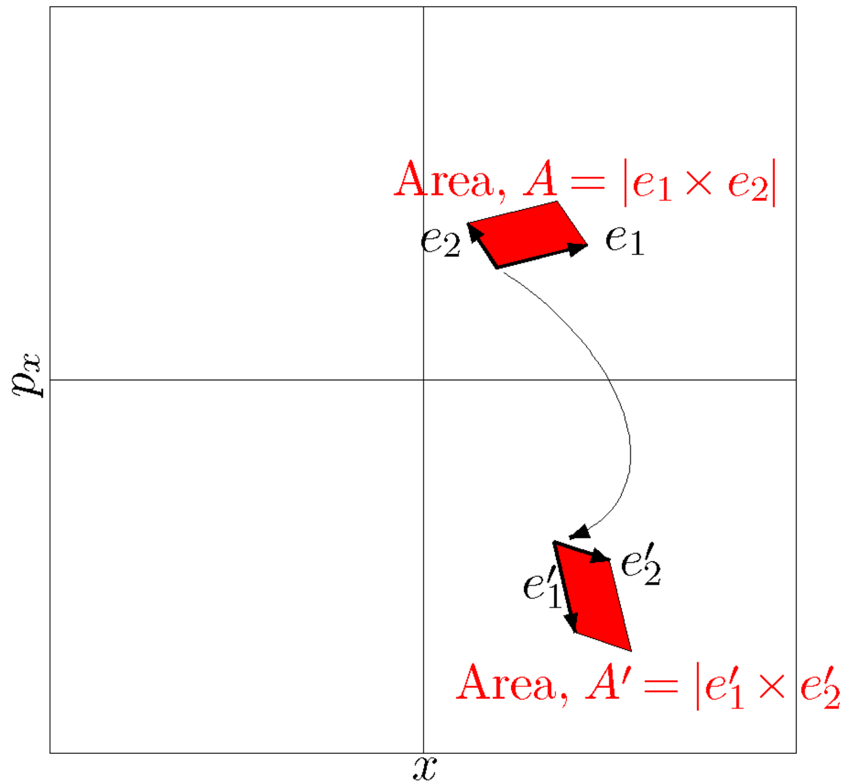
However, it is worth mentioning that the eigenvalues of the matrix:

$$\Sigma S = \langle \vec{x}\vec{x}^{\mathsf{T}} \rangle S \tag{51}$$

are conserved under a symplectic transformation.

The eigenvalues of ΣS are $\pm i\varepsilon_k$, where k is an index over the degrees of freedom of the system, and ε_k are the beam emittances.

Liouville's theorem is easiest to visualize in one degree of freedom, with a linear map...



...but the theorem generalizes to more degrees of freedom, and nonlinear maps.

As an example of a symplectic map, consider again the case of a drift space. To simplify things further, let us consider only the transverse motion. The map can be written:

$$X = x + \frac{p_x s}{\sqrt{1 - p_x^2}} \tag{52}$$

$$P_X = p_x. \tag{53}$$

The Jacobian is:

$$J = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial p_x} \\ \frac{\partial P_X}{\partial x} & \frac{\partial P_X}{\partial p_x} \end{pmatrix} = \begin{pmatrix} 1 & \frac{s}{(1-p_x^2)^{3/2}} \\ 0 & 1 \end{pmatrix}. \tag{54}$$

The Jacobian is a function of the dynamical variables; but we can still work out the matrix product with S. We find, as expected, that:

$$J^{\mathsf{T}} S J = S.$$

The case of three degrees of freedom starts to look more complicated, but we still find that the map is symplectic.

In our discussion so far, we considered transformations from one point along the reference trajectory to another.

However, we can also consider transformations that define a new set of variables $\vec{X}$ in terms of an existing set $\vec{x}$.

If the map is symplectic, then:

$$\vec{X} = \vec{X}(\vec{x}), \quad \text{where} \quad \frac{\partial \vec{X}}{\partial \vec{x}} = J, \quad \text{and} \quad J^{\mathsf{T}} S J = S. \tag{55}$$

If the original variables $\vec{x}$ are canonical (i.e. obey Hamilton's equations) then the new set of variables are also canonical:

$$\text{if} \quad \frac{d\vec{x}}{ds} = S \frac{\partial H}{\partial \vec{x}}, \quad \text{then} \quad \frac{d\vec{X}}{ds} = S \frac{\partial H}{\partial \vec{X}}. \tag{56}$$

A transformation from one set of canonical variables to another is called a *canonical transformation*.

Sometimes, it is convenient to work with dynamical variables other than the "cartesian" variables $(x, p_x, y, p_y, z, \delta)$. This is the case in nonlinear dynamics, where we often use "action–angle" variables.

The action–angle variables (J_x, ϕ_x) for the horizontal motion are defined by:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2, \tag{57}$$

$$\tan\phi_x = -\alpha_x - \beta_x \frac{p_x}{x}. \tag{58}$$

Here, α_x, β_x and γ_x are the usual Courant–Snyder parameters, defined for linear motion.

It can be shown that the Jacobian of the transformation is symplectic: therefore, (ϕ_x, J_x) are canonical variables. (Note that the angle ϕ_x is the co-ordinate, and the action J_x is the conjugate momentum).

Action–angle variables are useful for linear dynamics. In that case, we know that the betatron action is constant, and that the rate of increase of betatron phase is given by $1/\beta_x$:

$$\frac{d\phi_x}{ds} = \frac{1}{\beta_x}, \tag{59}$$

$$\frac{dJ_x}{ds} = 0. \tag{60}$$

Since action–angle variables are canonical variables, it should be possible to obtain these equations of motion from a suitable Hamiltonian. In fact, an appropriate Hamiltonian is given by:

$$H = \frac{J_x}{\beta_x}. \tag{61}$$

Summary

- The equations of motion for a particle moving through electromagnetic fields in an accelerator beamline (neglecting radiation and interactions between particles) can be derived from Hamilton's equations, with an appropriate Hamiltonian.

- Expressed in canonical variables, the transformation representing motion of a particle from one point along a beamline to another is symplectic (that is, the Jacobian of the transformation is a symplectic matrix).

- A symplectic transformation from one set of variables to another is called a canonical transformation. Sometimes, canonical transformations provide a way to simplify the equations of motion.

- An example of a canonical transformation is provided by the relationships between action–angle variables and the usual cartesian variables. Action-angle variables are widely used in accelerator beam dynamics.