Nonlinear Beam Dynamics

Part 1: Introduction

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Nonlinear effects play a crucial role both when designing an accelerator and during its operation.

Core Topics

- Transfer Maps
- Taylor Map
- Normal Forms
- Hamiltonian Mechanics
- Lie Transformations
- Symplectic Integration

H. Goldstein, "Classical Mechanics," Addison-Wesley (2nd edition, 1980).

A.J. Dragt, "Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics," (2009).
http://www.physics.umd.edu/dsat/dsatliemethods.html

E. Forest, "Beam Dynamics: A New Attitude and Framework," Taylor and Francis (1998).

A. Wolski, "Beam Dynamics in High Energy Particle Accelerators," Imperial College Press (2014).

W. Herr, "Mathematical and Numerical Methods for Non-linear Beam Dynamics," Proc. of the 2018 CERN-Accelerator-School.

H. Bartosik et al, "A first taste of nonlinear beam dynamics," Proc. of the 2021 CERN-Accelerator-School.

Any object creating nonlinear electromagnetic fields on the trajectory of the beam can strongly influence the beam dynamics.

These fields may arise from imperfections in the machine elements or be generated by the beam itself (space-charge, beam–beam interactions).

Nonlinear elements can be introduced deliberately into the lattice or may result from field errors in otherwise "linear" magnets. Both types must be taken into account for a reliable description of beam stability and dynamic aperture.

In the ideal case, dipole and quadrupole magnets produce pure $n = 1$ or $n = 2$ multipole fields, but systematic or random deviations introduce higher-order components.

In cylindrical coordinates $(r, \theta, s = 0)$ the transverse field components admit a multipole expansion

$$
\begin{aligned}
B_r(r,\theta) &= \sum_{n=1}^{\infty} \left[b_n \sin(n\theta) + a_n \cos(n\theta) \right] \left(\frac{r}{R_{\mathrm{ref}}} \right)^{n-1}, \\
B_\theta(r,\theta) &= \sum_{n=1}^{\infty} \left[b_n \cos(n\theta) - a_n \sin(n\theta) \right] \left(\frac{r}{R_{\mathrm{ref}}} \right)^{n-1},
\end{aligned}
\tag{1}
$$

where R_{ref} is a reference radius and b_n, a_n the normal and skew multipole coefficients. Equivalently, in Cartesian form with $z = x + iy = re^{i\theta}$,

$$
B(z) \;=\; \sum_{n=1}^{\infty} (b_n + i a_n) \left(\frac{z}{R_{\mathrm{ref}}} \right)^{n-1}. \tag{2}
$$

To correct chromaticity (momentum-dependent focusing) one introduces sextupoles of strength k_2.

These magnets generate fields larger than the intrinsic multipole errors of dipoles and quadrupoles and, if placed in regions of nonzero dispersion, provide an energy-dependent focusing kick.

In periodic lattices sextupole strengths can be kept small; in colliders special insertions with small dispersion and β-functions require stronger sextupoles, which in turn reduce the dynamic aperture.

Octupoles may be added to generate amplitude-dependent tune spread (Landau damping).

The beam itself produces nonlinear self-fields that perturb particle trajectories.

For a round beam of line density n and rms size σ, the radial electric and azimuthal magnetic fields are

$$\begin{aligned} E_r(r) &= -\frac{ne}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\int_0^\infty \frac{\exp\left(-r^2/(2\sigma^2+q)\right)}{2\sigma^2+q}\,dq, \\ B_\phi(r) &= -\frac{ne\,\beta c\,\mu_0}{4\pi}\frac{\partial}{\partial r}\int_0^\infty \frac{\exp\left(-r^2/(2\sigma^2+q)\right)}{2\sigma^2+q}\,dq. \end{aligned} \quad (3)$$

In colliding-beam machines these beam–beam forces require self-consistent treatment together with all other nonlinear fields in the ring.

In the standard approach to single-particle dynamics in rings one writes down the equations of motion and then seeks an ansatz to solve them.

For linear motion this leads to the well-known Courant–Snyder treatment, but it relies on assuming that the motion is both stable and bounded, which need not be known *a priori* in a non-linear system.

Instead of attempting to solve a global boundary-value problem, one may cast the problem as an initial-value problem by working directly with the *transfer maps* of the individual machine elements.

Consider the transverse coordinate $x(s)$ satisfying the Hill's equation:

$$\frac{d^2x}{ds^2} + K(s)\,x(s) \;=\; 0\,, \qquad (4)$$

where $K(s)$ is periodic in the ring (period C).

In that case one must solve a boundary-value problem $x(s+C) = x(s)$, which obscures generalization to non-periodic systems (linacs, beamlines, . . .).

Instead, for a linear, first order equation of the type

$$\frac{dx(s)}{ds} = K(s)x(s), \tag{5}$$

with an initial value at s_0, the solution can always be written as

$$\begin{aligned} x(s) &= a \cdot x(s_0) + b \cdot x'(s_0) \\ x'(s) &= c \cdot x(s_0) + d \cdot x'(s_0). \end{aligned} \tag{6}$$

This can be written in a matrix form as:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \tag{7}$$

In general, for each element spanning s_0 to s one defines the *linear transfer map*

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \mathcal{M}(s; s_0) \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}, \tag{8}$$

where $\mathcal{M}(s; s_0)$ is a 2×2 matrix of determinant one.

By construction this is an initial-value formulation, and the overall map of a lattice or ring is obtained by *composition* of the individual element maps:

$$\mathcal{M}(s_2; s_0) \;=\; \mathcal{M}(s_2; s_1)\, \mathcal{M}(s_1; s_0)\,. \tag{9}$$

Rather than solving a global boundary-value problem, one constructs the transfer map of the lattice by composing the maps $\mathcal{M}_i$ of individual elements. Starting from s_0,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+L} = \mathcal{M}_N \circ \mathcal{M}_{N-1} \circ \cdots \circ \mathcal{M}_1 \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \;\equiv\; \mathcal{M}(s_0, s_0 + L) \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}. \tag{10}$$

In a strictly periodic machine the most important map is the *One–Turn–Map* (OTM), which advances the phase-space vector $\mathbf{x} = (x, x')$ once around the ring.

If $\mathcal{M}(s_0, C)$ is the transfer map from s_0 to $s_0 + C$, then for any initial coordinates $(x(s_0), x'(s_0))$ we have

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \mathcal{M}_{\mathsf{OTM}} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}, \tag{11}$$

and the quadratic form

$$\begin{pmatrix} x \\ x' \end{pmatrix}^T_{s_0} \cdot \mathcal{M}_{\mathsf{OTM}} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = J \; = \; \text{const.} \tag{12}$$

is an invariant of the motion.

To include nonlinear fields one generalizes from matrices to multivariate Taylor maps.

For a single element one writes

$$\mathbf{x}_{\text{out}} = \mathcal{M}\Big(\mathbf{x}_{\text{in}}\Big) = \sum_{k_1,k_2,\ldots} M_{i,k_1 k_2 \cdots}\, x_{\text{in}}^{k_1}\, p_{x,\text{in}}^{k_2} \cdots , \tag{13}$$

where the coefficients $M_{i,k_1 k_2 \cdots}$ encode the nonlinear response up to any desired order.

The overall map is again obtained by composition (concatenation) of element maps, and may be symplectically truncated to finite order using techniques of *differential algebra*.

An alternative, manifestly symplectic representation uses Lie operators.

One seeks a factorization of the full one-turn (or one-pass) map in the form

$$\mathcal{M} = \exp\Big(:f_2:\Big)\ \exp\Big(:f_3:\Big)\ \exp\Big(:f_4:\Big)\ \cdots\ , \tag{14}$$

where each generator f_n is a homogeneous polynomial of degree n in the phase-space variables, and $:\cdot:$ denotes the Poisson-bracket operator.

In practice one computes the f_n using the Dragt–Finn factorization algorithm, yielding a sequence of symplectic maps truncated to any desired order.

In accelerator physics, symplecticity refers to the requirement that any map or integrator describing charged-particle motion must exactly preserve the phase-space volume and the underlying Hamiltonian structure.

Particle motion in electromagnetic fields is governed by a Hamiltonian, $H(q,p)$, so the true continuous flow $(q(t),p(t))$ conserves phase-space volume by Liouville's theorem.

A discrete transfer map, $\mathcal{M} : (q',p') \to (q,p)$ is symplectic if its Jacobian $J = \partial(q',p')/\partial(q,p)$ satisfies

$$J^T S J = S \tag{15}$$

where $S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and I is the identity in each degree of freedom.

Phase-space volume/area (emittance) remains constant under ideal transport—no artificial damping or growth.

Non-symplectic errors accumulate, producing spurious damping, excitation, or chaotic artifacts.

Ensures that numerical tracking faithfully reproduces resonances, dynamic-aperture limits, and tune shifts without unphysical artifacts.

Fundamental for accelerator design, as poor symplecticity can lead to erroneous predictions of beam loss or lifetime.

A linear map is symplectic if the matrix $\mathcal{M}$ representing the map is symplectic, i.e. satisfies:

$$\mathcal{M}^T S \mathcal{M} = S, \tag{16}$$

where, in one degree of freedom (i.e. two dynamical variables), S is the matrix:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{17}$$

In one degree of freedom, it is a necessary and sufficient condition for a matrix to be symplectic, that it has unit determinant: but this condition does *not* generalize to more degrees of freedom.

We shall consider what it means to say that a nonlinear map is symplectic later in this course.

In more degrees of freedom, S is constructed by repeating the 2×2 matrix above on the block diagonal, as often as necessary.

For example, map $\mathcal{M}$ from phase-space point $\mathbf{x}_1$ to $\mathbf{x}_2$ must be *symplectic*, meaning

$$\mathcal{M}^T \mathcal{S} \mathcal{M} = \mathcal{S}, \quad \mathcal{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{18}$$

so that the phase-space area is preserved.

The OTM $\mathcal{M}$ can be *diagonalized* into a *normal form* via a similarity transformation: find an invertible matrix A such that

$$A\,\mathcal{M}\,A^{-1} = R, \tag{19}$$

where R is as simple as possible (in the ideal case, a pure rotation). Equivalently,

$$\mathcal{M} = A\,R\,A^{-1}. \tag{20}$$

In this normal form, the motion on the phase-space ellipse is mapped to a rotation on a circle. Writing

$$A = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\alpha(s)/\sqrt{\beta(s)} & 1/\sqrt{\beta(s)} \end{pmatrix}, \quad R = \begin{pmatrix} \cos\Delta\mu & \sin\Delta\mu \\ -\sin\Delta\mu & \cos\Delta\mu \end{pmatrix}, \tag{21}$$

yields the Courant–Snyder (Twiss) parameters α, β and the phase advance $\Delta\mu$.

- The phase advance, μ_x is the "tune" $Q_x \cdot 2\pi$.

- β, α, ... are the optical/lattice parameters and describe phase space ellipse.

- The closed orbit (an invariant, identical coordinates after one turn): $\mathcal{M}_{\text{OTM}} \circ (x, x')_{\text{co}} = (x, x')_{\text{co}}$.

Upon transformation to normal form, it is natural to introduce *action–angle* variables (J_x, ψ_x) via

$$\begin{aligned} x &= \sqrt{2\,J_x\,\beta_x}\,\cos\psi_x, \\ p_x \equiv x' &= -\sqrt{\frac{2\,J_x}{\beta_x}}\Big(\sin\psi_x + \alpha_x\cos\psi_x\Big), \end{aligned} \tag{22}$$

where $J_x = \frac{1}{2}\Big(\gamma_x x^2 + 2\alpha_x p_x + \beta_x p_x^2\Big)$.

So that the motion corresponds to uniform rotation in ψ_x at constant radius $\sqrt{2J_x}$ in the normalized phase space.

For a beam (ensemble of particles), the rms emittance ε_x is defined as the average action:

$$\langle x^2 \rangle = \langle 2\, J_x\, \beta_x \cos^2 \psi_x \rangle = 2\, \beta_x \,\langle J_x \rangle \langle \cos^2 \psi_x \rangle \tag{23}$$

using $\langle \cos^2 \psi_x \rangle = \frac{1}{2}$.

One also finds

$$\langle p_x^2 \rangle = \gamma_x\, \varepsilon_x, \qquad \langle x\, p_x \rangle = -\alpha_x\, \varepsilon_x, \tag{24}$$

and hence the emittance can be written in terms of second moments as

$$\varepsilon_x = \sqrt{\langle x^2 \rangle\, \langle p_x^2 \rangle - \langle x\, p_x \rangle^2}. \tag{25}$$

Any smooth non-linear map can be expanded in a truncated Taylor series.

In 2D transverse phase space,

$$x_j(s_2) = \sum_{k=1}^{4} R_{jk}\, x_k(s_1) + \sum_{k=1}^{4}\sum_{l=1}^{4} T_{jkl}\, x_k(s_1)\, x_l(s_1), \tag{26}$$

where x_j for $j = 1, \ldots, 4$ represents (x, x', y, y') and (R, T) define the 2nd-order map $\mathcal{M}^{(2)}$; higher orders add tensors, e.g., for the 3rd order map $\mathcal{M}^{(3)} = (R, T, U)$ we add a third order tensor:

$$+ \sum_{k=1}^{4}\sum_{l=1}^{4}\sum_{m=1}^{4} U_{jklm} x_k(s_1)\, x_l(s_1)\, x_m(s_1). \tag{27}$$

In order to be a symplectic transfer map, the Jacobian matrix $\mathcal{J}$ should fulfill the symplectic condition:

$$J^T S\, J = S \tag{28}$$

where $J_{jk} = \partial x_j(s_2)/\partial x_k(s_1)$.

This generally forces relations among the Taylor coefficients.

The explicit map for a sextupole is:

$$
\begin{aligned}
x_2 &= x_1 + L x_1' \quad -k_2 \Big[\frac{L^2}{4}(x_1^2 - y_1^2) + \frac{L^3}{12}(x_1 x_1' - y_1 y_1') + \frac{L^4}{24}(x_1'^2 - y_1'^2) \Big] \\
x_2{}' &= x_1' \qquad\qquad -k_2 \Big[\frac{L}{2}(x_1^2 - y_1^2) + \frac{L^2}{4}(x_1 x_1' - y_1 y_1') + \frac{L^3}{6}(x_1'^2 - y_1'^2) \Big] \\
y_2 &= y_1 + L y_1' \qquad\qquad +k_2 \Big[\frac{L^2}{4} x_1 y_1 + \frac{L^3}{12}((x_1 y_1' + y_1 x_1') + \frac{L^4}{24} x_1' y_1' \Big] \\
y_2' &= y_1' \qquad\qquad\qquad +k_2 \Big[\frac{L}{2} x_1 y_1 + \frac{L^2}{4}((x_1 y_1' + y_1 x_1') + \frac{L^3}{6} x_1' y_1' \Big]
\end{aligned}
\qquad (29)
$$

Then, the Jacobian matrix is:

$$
J = \begin{pmatrix}
\frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_1'} \\
\frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_1'} & \frac{\partial x_2'}{\partial y_1} & \frac{\partial x_2'}{\partial y_1'} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_1'} & \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_1'} \\
\frac{\partial y_2'}{\partial x_1} & \frac{\partial y_2'}{\partial x_1'} & \frac{\partial y_2'}{\partial y_1} & \frac{\partial y_2'}{\partial y_1'}
\end{pmatrix}
\qquad (30)
$$

Symplecticity of the Taylor Map: Sextupole

For example,

$$\begin{pmatrix} \frac{\partial x_2}{\partial x_1} = 1 - k_2\left(\frac{L^2}{2}x_1 + \frac{L^3}{12}x_1'\right) & \frac{\partial x_2}{\partial x_1'} = L - k_2\left(\frac{L^3}{12}x_1 + \frac{L^4}{12}x_1'\right) \\ \frac{\partial x_2'}{\partial x_1} = -k_2\left(Lx_1 + \frac{L^2}{4}x_1'\right) & \frac{\partial x_2'}{\partial x_1'} = 1 - k_2\left(\frac{L^2}{4}x_1 + \frac{L^3}{3}x_1'\right) \end{pmatrix} \tag{31}$$

For $k_2 = 0$,

$$\begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

For non-zero k_2, the elements of Jacobian matrix depend on initial values, i.e., (x_1, y_1).

Therefore, the Taylor transfer map is not symplectic.

Exact maps for *thick* elements often lack closed-form solutions.

Instead, one uses *thin*-lens kicks interleaved with drifts, i.e., symplecticfying by construction, and refines accuracy by slicing longer elements into multiple kicks.

A transfer map for a thick, linearized quadrupole of length L and strength K:

$$\mathcal{M} = \begin{pmatrix} \cos(L\sqrt{K}) & \frac{1}{\sqrt{K}}\sin(L\sqrt{K}) \\ -\sqrt{K}\sin(L\sqrt{K}) & \cos(L\sqrt{K}). \end{pmatrix} \quad (\det\mathcal{M} = 1), \quad (33)$$

For a "small" length L, the transfer map of a quadrupole can be extended as a Taylor series:

$$\mathcal{M}_{s\to s+L} = L^0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1 \cdot \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} + L^2 \cdot \begin{pmatrix} -\frac{1}{2}K & 0 \\ 0 & -\frac{1}{2}K \end{pmatrix} + \cdots \tag{34}$$

If we keep terms up to the first order of L, then

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1 & L \\ -KL & 1 \end{pmatrix} + \mathcal{O}(L^2) \tag{35}$$

This transfer map is exact to the order of $\mathcal{O}(L^2)$, but this truncated map is not *symplectic*.

Adding an $O(L^2)$ correction restores $\det \mathcal{M} = 1$ without degrading accuracy:

$$\mathcal{M} = \begin{pmatrix} 1 & L \\ -KL & 1 - KL^2 \end{pmatrix} + O(L^2). \tag{36}$$

This transfer map can be obtained with the drift followed by the kick:

$$\begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & L \\ -KL & 1 - KL^2 \end{pmatrix} \tag{37}$$

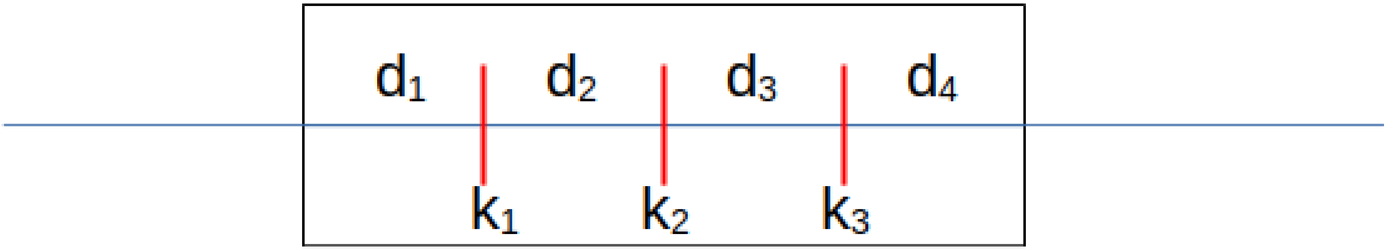
We can construct a symplectic map to the order of $\mathcal{O}(L^2)$ with a single kick of strength KL at the element's center sandwiched by drifts $L/2$ each.

$$\begin{aligned}\mathcal{M}_{s\to s+L} &= \begin{pmatrix}1 & \frac{1}{2}L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -KL & 1\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2}L\\ 0 & 1\end{pmatrix} + \mathcal{O}(L^2)\\ &\approx \begin{pmatrix}1-\frac{1}{2}KL^2 & L-\frac{1}{4}KL^3\\ -KL & 1-\frac{1}{2}KL^2\end{pmatrix}\end{aligned} \tag{38}$$

Using a drift–kick–drift model (kick in the middle) yields an $O(L^2)$ integrator, whereas applying the kick only at the entry or exit gives $O(L^1)$ accuracy.

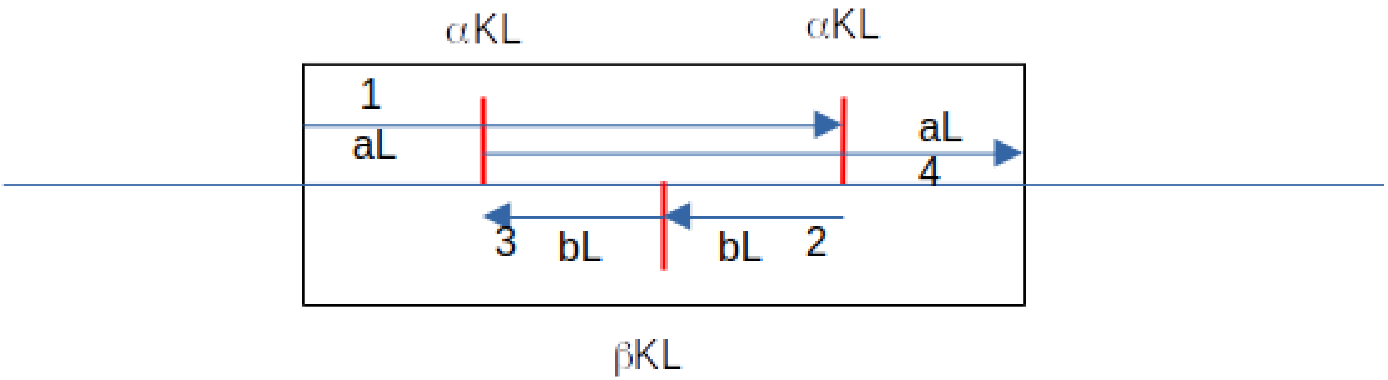
Thus proper splitting improves both accuracy and symplectic consistency.

Assume that an element is split into three kicks with 4 drift spaces:



The optimized strengths and drifts spaces between three kicks are:

$$\begin{aligned} a &\approx 0.6756 \quad & b &\approx -0.1756 \\ \alpha &\approx 1.3512 \quad & \beta &\approx -1.7024 \end{aligned} \tag{39}$$



Ref) E. Forest *et al*, "Fourth-order symplectic integration," Physica D 43 (1990) 105.

Higher Order Symplectic Integration: Yoshida Method

From the previous example, we learned that a fourth order symplecitc integrator (3-kicks) can be constructed with a second order symplectic opeartor (1-kick):

$$S_4(t) = S_2(x_1 t) \circ S_2(x_0 t) \circ S_2(x_1 t) \tag{40}$$

where $x_0 = \frac{-2^{1/3}}{2-2^{1/3}} \approx -1.7024$ and $x_1 = \frac{1}{2-2^{1/3}} \approx 1.3512$.

Then, from a fourth order to a sixth order:

$$S_6(t) = S_4(x_1 t) \circ S_4(x_0 t) \circ S_4(x_1 t) \tag{41}$$

Therefore, in general,

$$S_{k+2}(t) = S_k(x_1 t) \circ S_k(x_0 t) \circ S_k(x_1 t) \tag{42}$$

Ref) H. Yoshida, "Construction of higher order symplectic integrators," Phys. Lett. A, 150 (1990), p. 262.

Small periodic perturbations combine coherently when the tunes (ν_x, ν_y) satisfy a resonance condition

$$m_x \nu_x + m_y \nu_y = \ell, \tag{43}$$

with integers m_x, m_y, ℓ.

The order of the resonance is $|m_x| + |m_y|$ (first order: integer; second: half-integer; etc.).

Resonances may be driven by dipole, quadrupole or higher-order multipole fields, including sextupoles.

- *Dipole errors:* A kick from a dipole-field error at $s = s_0$ produces an angular deflection $\Delta x'$. After one cell of phase advance μ_x, the net kick over N identical cells is

$$\sum_{k=0}^{N-1} \Delta x' \cos(k\mu_x) = \Delta x' \frac{\sin\Big(N\mu_x/2\Big)}{\sin(\mu_x/2)} \cos\Big(\tfrac{(N-1)\mu_x}{2}\Big).$$

  This vanishes if $\mu_x = \pi$ (half-integer tune, $\nu_x = \mu_x/2\pi = 1/2$) and peaks if $\mu_x = 2\pi$ ($\nu_x = 1$).

- *Quadrupole errors:* A focusing-error kick $\Delta k\, x$ similarly adds coherently when $\mu_x = \pi$ (half-integer tune) and cancels at $\mu_x = 2\pi$.

- *Sextupole kicks:* A thin sextupole kick $\Delta p_x = -\frac{1}{2}k_2 L\, x^2$ behaves like a second-order perturbation. The kicks from successive cells cancel if $\mu_x = (2m+1)\pi$ (ν_x= half-integer) and reinforce if $\mu_x = 2m\pi$ (ν_x= integer).

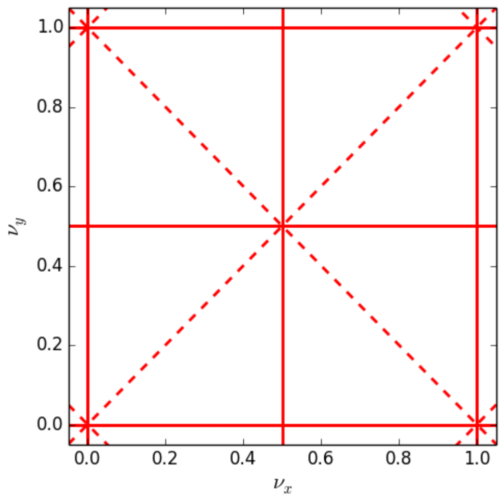
Particles experience resonant instability whenever their horizontal and vertical tunes (ν_x, ν_y) satisfy

$$m_x \nu_x + m_y \nu_y = \ell, \quad m_x, m_y, \ell \in \mathbb{Z}. \tag{79}$$

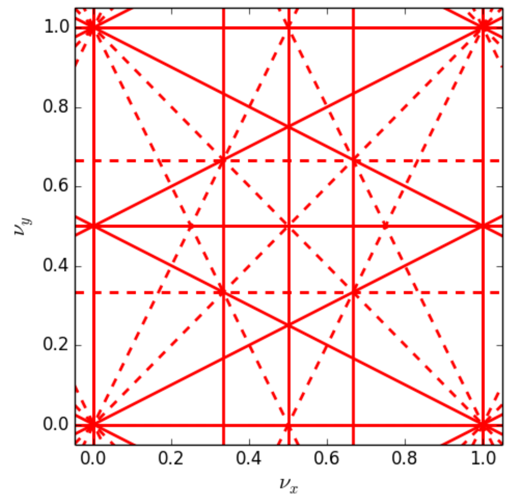
- The *order* of the resonance is $|m_x| + |m_y|$. - Examples:

- Integer resonance ($|m_x| + |m_y| = 1$), e.g. $\nu_x = 1$.
- Half-integer ($|m_x| + |m_y| = 2$), e.g. $2\nu_x = 1$ or $\nu_x + \nu_y = 1$.
- Third-order ($|m_x| + |m_y| = 3$) can be driven by sextupoles.

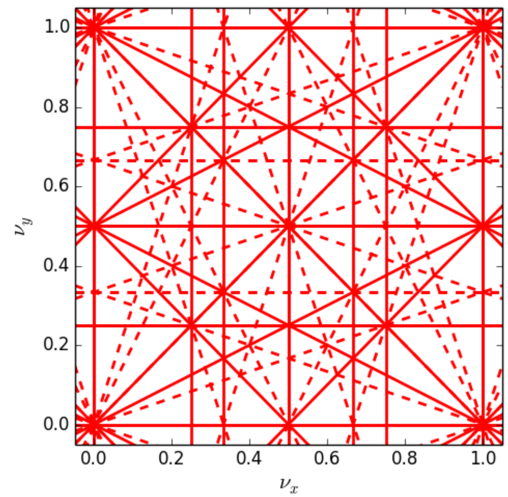
Tune diagram



a) up to second order



b) up to third order



c) up to fourth order

Although one often associates a resonance of order n with an nth-order multipole (e.g. sextupole $\to n = 3$), in practice:

- Higher-order multipoles can contribute to lower- and higher-order resonances via nonlinear coupling.

- The actual strength of a particular (m_x, m_y) resonance depends on the Fourier coefficients of the combined perturbations along the lattice.

For a ring built from P identical cells, the resonance ℓ is suppressed ("non-systematic") at first order if

$$\frac{\ell}{P} \notin \mathbb{Z},$$

because the phase advance per cell $\mu_x = 2\pi\nu_x$ satisfies

$$m_x \frac{\nu_x}{P} + m_y \frac{\nu_y}{P} = \frac{\ell}{P}$$

which is non-integer and causes cancellation of kicks over one turn.

If $\ell/P \in \mathbb{Z}$, the resonance is *systematic* and kicks add coherently.

- Ideal symmetry ($P > 1$) protects against certain resonances.

- Real-machine errors reduce effective $P \to 1$, making all resonances systematic.

- Choose working point (ν_x, ν_y) away from low-order systematic resonances.

- Use lattice periodicity and symmetry to cancel harmful resonances.

- Employ harmonic sextupoles/octupoles to compensate residual driving terms.