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Synchrotron radiation

(Part 2)

If synchrotron radiation were a purely classical process, then the beam emittances would damp to zero. However, it is observed in practice that the emittances reach non-zero equilibrium values, which depend on Planck's constant ($\hbar$).

1. Radiation is not emitted continuously by a particle in a magnetic field, but in discrete quanta (photons). The random emission of photons acts as a 'noise' term in the equations of motion for a particle, which leads to some excitation of betatron (horizontal only) and synchrotron oscillations.

2. Since there is a small opening angle (approximately $1/\gamma$), there is a small, but non-zero lower limit on the vertical emittance as well. In practice, though, the vertical emittance in storage rings is dominated by machine errors (such as magnet alignment errors) that introduce small amounts of vertical dispersion and betatron coupling.

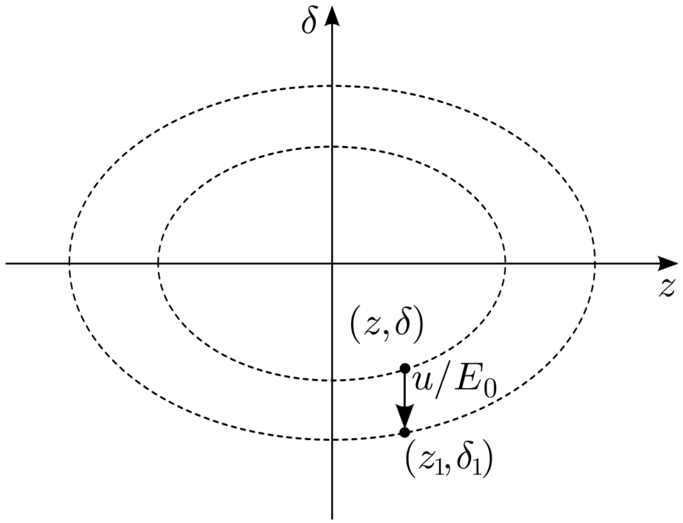
3. The rate of excitation depends on the rate of emission of photons and on the photon energy distribution, and is not dependent on the amplitude of the synchrotron or betatron oscillations already being performed by an electron. Therefore, the emission of photons does not lead to an exponential increase in the oscillation amplitude, but rather to a linear increase (as a function of time). This means that at some particular value of the emittance, the rate of increase is matched by the rate of damping.

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x\tau_x}C_q\gamma^2\frac{I_5}{I_2}, \tag{1}$$

$$\frac{d\sigma_\delta^2}{dt} = -\frac{2}{\tau_z}\sigma_\delta^2 + \frac{2}{j_z\tau_z}C_q\gamma^2\frac{I_3}{I_2}. \tag{2}$$

Suppose an electron emits a photon of energy u: this leads to an instantaneous change in the energy deviation δ. Using Eq. (3.17) in p.3-9, the values of z and δ following the photon emission are

$$z' = z \simeq \frac{\alpha_p c}{\omega_s}\delta_0 \cos(\theta), \quad \delta' = \delta - \Delta\delta = \delta_0 \sin(\theta) - \frac{u}{E_0}. \qquad (3)$$



We consider the case with $\eta_p \simeq \alpha_p > 0$ (above transition for $\gamma \gg 1$). Hence, particle is moving in CCW direction, and θ is negative for the dots in the figure above.

Using,

$$\frac{\alpha_p c}{\omega_s}\delta_0 \cos(\theta) = \frac{\alpha_p c}{\omega_s}\delta_0' \cos(\theta'), \quad \delta_0 \sin(\theta) - \frac{u}{E_0} = \delta_0' \sin(\theta'), \tag{4}$$

the change in the amplitude of the energy oscillation can then be found from

$$\delta_0'^2 = \delta_0^2 - 2\delta_0 \frac{u}{E_0} \sin(\theta) + \frac{u^2}{E_0^2}. \tag{5}$$

Averaging over all particles in the bunch gives the change in the mean square energy deviation:

$$\begin{aligned} \Delta\sigma_\delta^2 &= \langle \delta'^2 \rangle - \langle \delta^2 \rangle \\ &= \langle \delta_0'^2 \sin^2(\theta') \rangle - \langle \delta_0^2 \sin^2(\theta) \rangle \\ &= \frac{\langle u^2 \rangle}{2E_0^2}. \end{aligned} \tag{6}$$

Suppose that photons in the energy range u to $u + du$ are emitted at a rate $\dot{N}(u)$.

$$\frac{d\langle u^2\rangle}{dt} = \int_0^\infty \dot{N}(u)u^2 du. \tag{7}$$

If we include radiation damping as well as the excitation from photon emission, the overall rate of change of the mean square energy deviation is

$$\frac{d\sigma_\delta^2}{dt} = \frac{1}{2E_0^2}\left\langle \int_0^\infty \dot{N}(u)u^2 du \right\rangle_C - \frac{2}{\tau_z}\sigma_\delta^2, \tag{8}$$

where$\langle \ldots \rangle_C$ indicates an average over the circumference of the ring.

The rate of photon emission $\dot{N}(u)$ can be calculated from the synchrotron radiation spectrum using the fact that the energy of a single photon is $u = \hbar\omega$ [Jackson Chap. 14 (1998), Sands (1970)].

$$\int_0^\infty \dot{N}(u) u^2 du = 2C_q \gamma^2 \frac{E_0}{\rho} P_\gamma. \tag{9}$$

Here, C_q is the synchrotron radiation quantum constant:

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc}. \tag{10}$$

Finally, we find

$$\frac{d\sigma_\delta^2}{dt} = \frac{2}{j_z \tau_z} C_q \gamma^2 \frac{I_3}{I_2} - \frac{2}{\tau_z} \sigma_\delta^2, \tag{11}$$

where I_3 is the third synchrotron radiation integral:

$$I_3 = \oint \frac{1}{|\rho|^3} ds. \tag{12}$$

A balance between quantum excitation and radiation damping is reached when

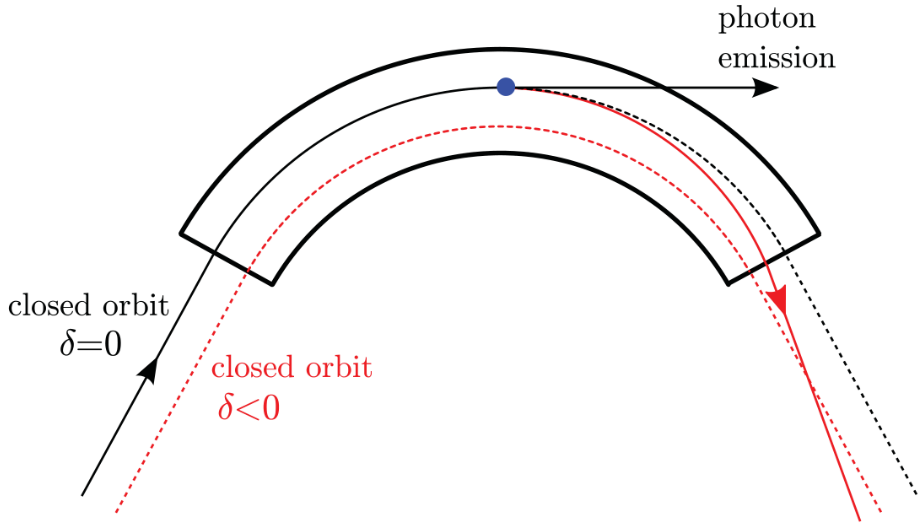
$$\sigma_\delta^2 = \sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \tag{13}$$

which is known as the natural energy spread.

In practice, the actual energy spread may be larger than the natural energy spread, because collective effects (interactions between the electrons) can cause the energy spread to increase.

Since the ratio of the amplitude of the co-ordinate oscillation to the amplitude of the energy oscillation is $\eta_p c/\omega_s$, the natural bunch length associated with the natural energy spread is

$$\sigma_{z0} = \frac{|\eta_p| c}{\omega_s} \sigma_{\delta 0} \propto \sqrt{|\eta_p|}. \tag{14}$$



If a particle with $\delta = 0$ is initially following a closed orbit (black line), then, following the emission of a photon, there is a change in the closed orbit described by the dispersion (i.e., $\Delta x = \eta_x \delta < 0$).

Since there is an instantaneous loss of energy, the particle (now with negative energy deviation, $\delta < 0$) is no longer on a closed orbit. As a result of the photon emission, the particle follows a trajectory (solid red line) in which it makes betatron oscillations around the new closed orbit (dashed red line).

A full analysis (see, Sands[1970]) leads to the result for the rate of change of the horizontal emittance:

$$\frac{d\varepsilon_x}{dt} = \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2} - \frac{2}{\tau_x} \varepsilon_x, \tag{15}$$

where I_5 is the fifth synchrotron radiation integral given by

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|} ds, \tag{16}$$

and the dispersion $\mathcal{H}$-function (sometimes called the 'curly-H' function) is

$$\mathcal{H} = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_x' + \beta_x \eta_x'^2. \tag{17}$$

The natural emittance is

$$\varepsilon_x = \varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}. \tag{18}$$

Storage rings for 3rd-generation synchrotron light sources typically operate with natural emittances of order of a few nm (e.g, ~5.8 nm for PLS-II).

- Double-bend achromat (DBA):
- Dispersion leak:

For 4th-generation synchrotron light sources (e.g, ~62 pm for Korea-4GSR),

- Multi-bend achromat (MBA):
- Longitudinal gradient bending magnet (LGBM):
- Reverse bend (RB):

Without vertical dispersion and betatron coupling, the corresponding quantity to I_5 in the vertical direction will be zero, which suggests that the equilibrium vertical emittance will also be zero.

However, in fact, there is some radiation emitted at (small) angles above and below the horizontal plane ($\sim 1/\gamma$), which introduces vertical momentum to the particle.

Then, the equilibrium vertical emittance in a ring with no vertical dispersion or betatron coupling is given by [Raubenheimer (1991)]

$$\varepsilon_y = \frac{13}{55}\frac{C_q}{I_2}\oint \frac{\beta_y}{|\rho^3|}ds \;\sim\; \text{order 0.1 pm} \;<\; \frac{\lambda}{4\pi}. \tag{19}$$

However, even a small amount of vertical dispersion or betatron coupling, from random alignment errors in the magnets, can lead to vertical emittances of order of tens of pm. (If the vertical emittance is too small, Touschek scattering causes loss of particles. So, tens of pm is even better.)

[General remarks]

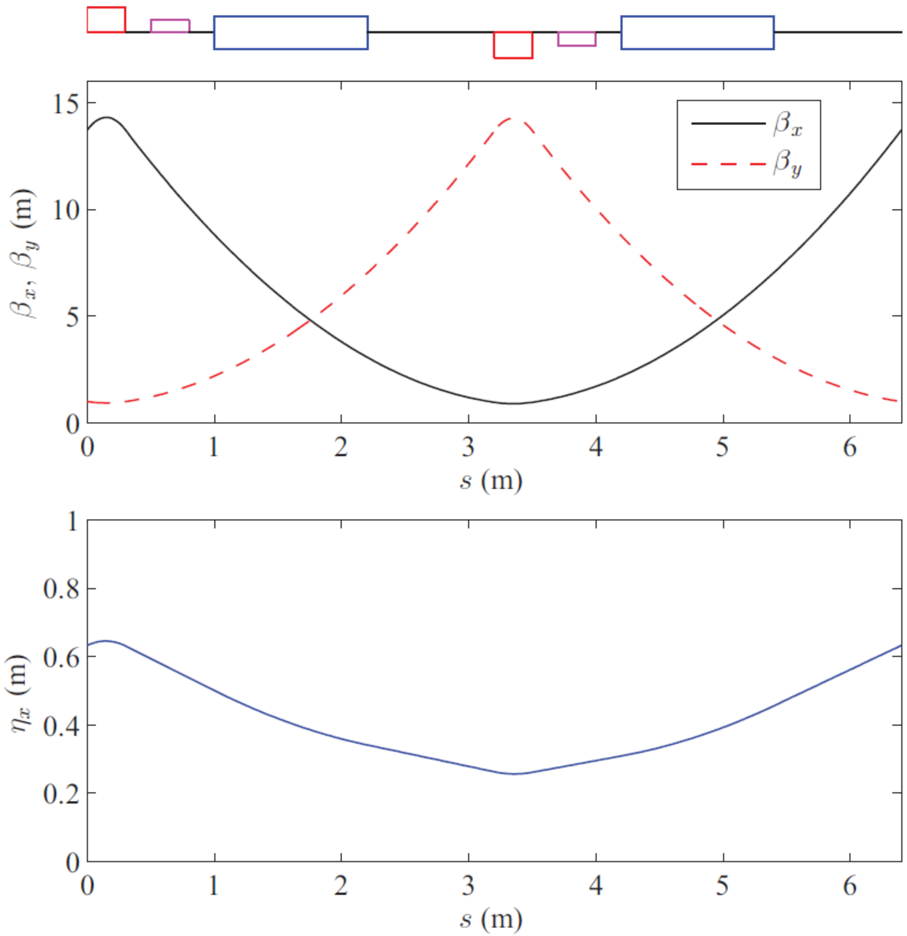
It is important to achieve a small value for the natural emittance: this helps to produce synchrotron radiation with high brightness.

The natural emittance depends on the beam energy ($\propto \gamma^2$) and the ratio $I_5/j_x I_2$.

Usually, the beam energy is determined by considerations such as the wavelength range that the synchrotron radiation should cover. With the curvature of the beam trajectory limited by magnet technology, a high beam energy is needed to generate short wavelength synchrotron radiation.

The ratio $I_5/j_x I_2$ is determined entirely by the radius of curvature and quadrupole field component (in the dipole magnets and IDs), and by the optical functions (i.e. the Courant–Snyder parameters and the dispersion).

The simplest structure for the magnetic lattice in a storage ring is based on the FODO cell, consisting of alternating focusing (F) and defocusing (D) quadrupoles, with drift spaces (O) or dipole magnets between the quadrupoles.



Consider a FODO cell in which the quadrupole magnets (as thin lenses) have focal length $\pm f$, and the space between the quadrupoles is entirely occupied by dipole magnets of length $L = \rho\theta$.

The transfer matrix for a thin quadrupole of focal length f is

$$R_{\text{quad}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}. \tag{20}$$

The horizontal part of the transfer matrix through a dipole of bending angle θ is

$$R_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}. \tag{21}$$

Using the results of Chapter 2, we can calculate the Courant-Snyder parameters and the dispersion through the FODO cell.

For example, at the horizontally-focusing quadrupole

$$\alpha_x = 0, \quad \beta_x = \frac{4f\rho\sin\theta(2f\cos\theta + \rho\sin\theta)}{\sqrt{16f^4 - [\rho^2 - (4f^2 + \rho^2)\cos 2\theta]^2}}, \tag{22}$$

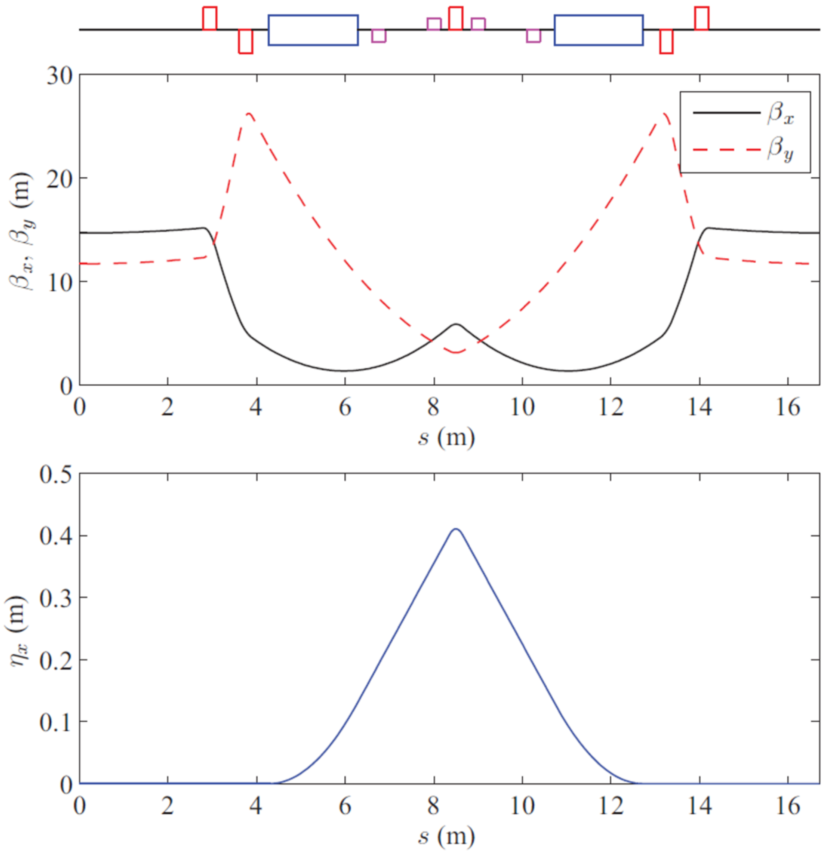
We can then perform the integral I_5, and express it in terms of the FODO cell parameters. (Warning: Do not attempt it by hand; use Mathematica or similar tools instead. The algebra is formidable.) The result can be expressed as a power series in θ:

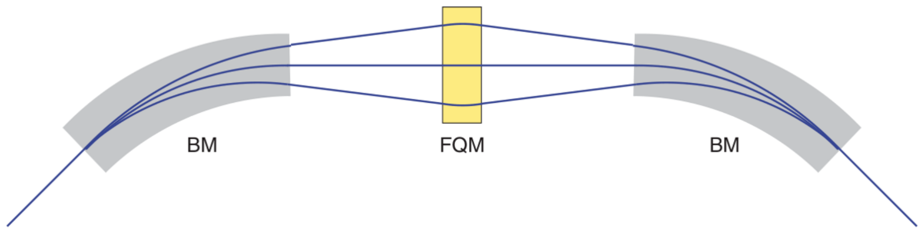
$$\frac{I_5}{I_2} = \left(4 + \frac{\rho^2}{f^2}\right)^{-\frac{3}{2}} \left[8 - \frac{\rho^2}{2f^2}\theta^2 + O(\theta^4)\right]. \tag{23}$$

In particular, assuming that $\rho \gg 2f \gg L/2$, and using approximation $j_x \approx 1$,

$$\varepsilon_{0,\mathrm{FODO}} \approx C_q\gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3. \tag{24}$$

To achieve a smaller emittance than in a FODO ring, we consider a more sophisticated cell structure such as the double-bend achromat (DBA). The central region of a DBA cell consists of a pair of dipole magnets with a quadrupole magnet placed midway between them.





The first dipole magnet generates dispersion.

The strength of the quadrupole is chosen to reverse the gradient of the dispersion; that is, if the gradient of the dispersion is $\eta_x' = d\eta_x/ds$ at the entrance of the quadrupole, then at the exit of the quadrupole the gradient of the dispersion is $-\eta_x'$.

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_x' \end{pmatrix} = \begin{pmatrix} \eta_x \\ -\eta_x' \end{pmatrix} \tag{25}$$

In that case, by the symmetry, the second dipole magnet exactly cancels the dispersion generated by the first dipole magnet. Hence, in a DBA cell, dispersion is present only in the region between the dipole magnets.

The cell is completed by additional quadrupoles outside of the pair of dipole magnets (i.e. in the region with zero dispersion), which can be used to adjust the Courant–Snyder parameters and phase advance.

The benefit of a DBA cell is that the dispersion can be kept much smaller in the dipole magnets than is the case in a FODO cell. As a result, $I_5/j_x I_2$ can be smaller than in a FODO cell, for the same dipole magnet parameters. After some complicated algebra,

$$\varepsilon_{0,\mathrm{DBA}} \approx \frac{C_q}{4\sqrt{15}}\gamma^2\theta^3. \tag{26}$$

A further benefit of the DBA lattice style is that it naturally provides zero-dispersion sections at regular intervals around the ring; these sections can be made (in principle) of any desired length, and are ideal locations for IDs.

If an ID is placed at a location with non-zero dispersion, then quantum excitation in the ID leads to an increase in the natural emittance. If the dispersion is reasonably small, this may not be a significant effect.

A DBA cell is ideally suited to a (reasonably) low-emittance synchrotron needing a large number of IDs to serve many users.

In principle, it is possible to achieve a lower natural emittance than that in a DBA by allowing non-zero dispersion throughout the cell: this removes the constraint that $\eta_x = \eta_x' = 0$ at the entrance of the first dipole magnet in the cell, and at the exit of the second dipole.

It is then possible to optimise the Courant–Snyder parameters and the dispersion to minimise the value of $I_5/j_x I_2$ in the cell. The resulting structure is known as a theoretical minimum emittance (TME) lattice.

A TME cell consists of a single dipole, with quadrupoles to control the Courant-Snyder parameters and dispersion through the cell.

The natural emittance in a TME storage ring is given by (after complicated algebra)

$$\varepsilon_{0,\mathrm{TME}} \approx \frac{C_q}{12\sqrt{15}} \gamma^2 \theta^3. \qquad (27)$$

However, the TME cell has a number of drawbacks.

- In particular, dispersion is nonzero throughout a TME lattice, so we lose the advantage of the DBA cell in providing locations with zero (or relatively small) dispersion for IDs.

- Also, nonlinear effects in the beam dynamics can be very strong in a TME lattice. Correction of chromaticity can be very difficult, and the dynamic aperture tends to be very small, which leads to a reduction in beam lifetime.

It is possible to improve on the performance offered by a DBA lattice by adopting some of the features of a TME cell.

We detune the quadrupole between each pair of dipoles and thereby allow some dispersion to leak in the region in the (nominally) zero-dispersion regions.

Although dispersion now occurs throughout both dipoles, the integral of the curly-H function $\mathcal{H}$ can actually be reduced in this way, leading to a reduction in the natural emittance.

While there is then some dispersion at locations of IDs, if the dispersion is carefully controlled then the resulting increase of emittance from quantum excitation in the IDs can be limited.

[Note] Including a (transverse) gradient in the dipole field provides an extra degree of freedom to reduce the natural emittance, either by reducing I_5 or by increasing j_x.

Another way to improve on the DBA cell is to include more dipoles within each cell; the cell is then known as a multi-bend achromat (MBA).

The advantage of the MBA structure is that it allows the dispersion to be adjusted to zero at either end of the cell, while the dispersion and Courant–Snyder parameters are tuned to be close to the TME conditions in the dipole magnets in the middle of the cell.

The minimum natural emittance in a MBA with M dipoles per cell is (after complicated algebra)

$$\varepsilon_{0,\mathrm{MBA}} \approx \frac{C_q}{12\sqrt{15}} \left(\frac{M+1}{M-1} \right) \gamma^2 \theta^3. \tag{28}$$

Here, θ is the average bending angle per dipole.

- Keeping the bending angle per dipole constant, the ratio between the natural emittances of the MBA and DBA is

$$\frac{\varepsilon_{0,\mathrm{MBA}}}{\varepsilon_{0,\mathrm{DBA}}} = \frac{1}{3}\left(\frac{M+1}{M-1}\right). \tag{29}$$

- For $M = 2$, $\varepsilon_{0,\mathrm{MBA}} \to \varepsilon_{0,\mathrm{DBA}}$, and for $M \to \infty$, $\varepsilon_{0,\mathrm{MBA}} \to \varepsilon_{0,\mathrm{DBA}}/3 = \varepsilon_{0,\mathrm{TME}}$

- If we keep the ring circumference more or less the same (with a given energy), then the bending angle per dipole is

$$\frac{\theta_{\mathrm{MBA}}}{\theta_{\mathrm{DBA}}} \sim \frac{2}{M}. \tag{30}$$

$$\frac{\varepsilon_{0,\mathrm{MBA}}}{\varepsilon_{0,\mathrm{DBA}}} \sim \frac{1}{3}\left(\frac{M+1}{M-1}\right)\left(\frac{2}{M}\right)^3. \tag{31}$$

For $M = 7$, $\varepsilon_{0,\mathrm{MBA}}/\varepsilon_{0,\mathrm{DBA}} \sim 32/3082 \sim 1/100$.

[1] Introduction to Beam Dynamics in High-Energy Electron Storage Rings, Andrzej Wolski.

[2] Beam Dynamics In High Energy Particle Accelerators, Andrzej Wolski.

[3] An Introduction to Synchrotron Radiation: Techniques and Applications, Philip Willmott.