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Synchrotron radiation

(Part 1)

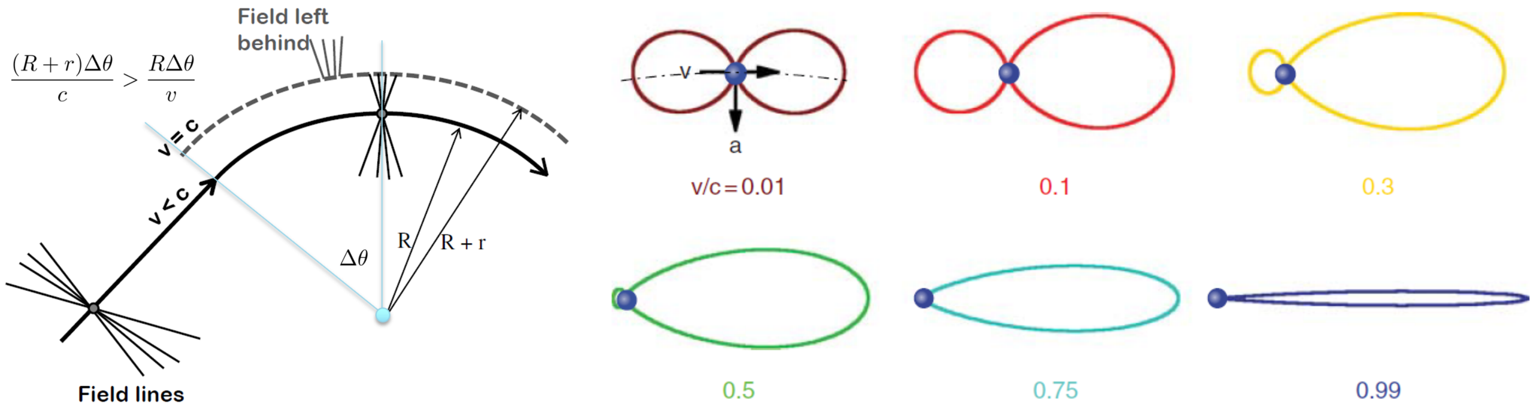
Any acceleration of a charged particle leads to disturbances in the fields around the particle, and hence produces EM radiation. If the particle is moving relativistically, the radiation is known as synchrotron radiation.

In an electron storage ring, particles accelerate (by changing their direction of motion, though not their speed) as they move through the magnetic fields in the ring.

- In quadrupole (and higher-order multipole) magnets, the fields seen by particles are generally much weaker than in the dipole magnets, so the amount of synchrotron radiation produced in these magnets is (except in some special cases) negligible.
- In a linac, the rate of acceleration is small at ultrarelativistic velocities, so the amount of radiation is also small.

The amount of synchrotron radiation produced by accelerating a charged particle depends on the charge-to-mass ratio of the particle as well as on the rate of acceleration.

Although synchrotron radiation is produced by protons in storage rings, since protons have much larger mass than electrons, the amount of radiation from protons is small, and can usually be ignored.



One of the main effects of synchrotron radiation is to lead to the beam reaching equilibrium emittances, determined by the beam energy and the arrangement of magnets.

A (ultra-relativistic) particle ($\beta \approx 1$) with energy E following a curved trajectory with radius ρ emits radiation with power

$$P_\gamma = \frac{C_\gamma}{2\pi}\frac{cE^4}{\rho^2} = \frac{C_\gamma}{2\pi}q^2c^3E^2B^2. \tag{1}$$

C_γ is a constant given by

$$C_\gamma = \frac{q^2}{3\epsilon_0(mc^2)^4}, \tag{2}$$

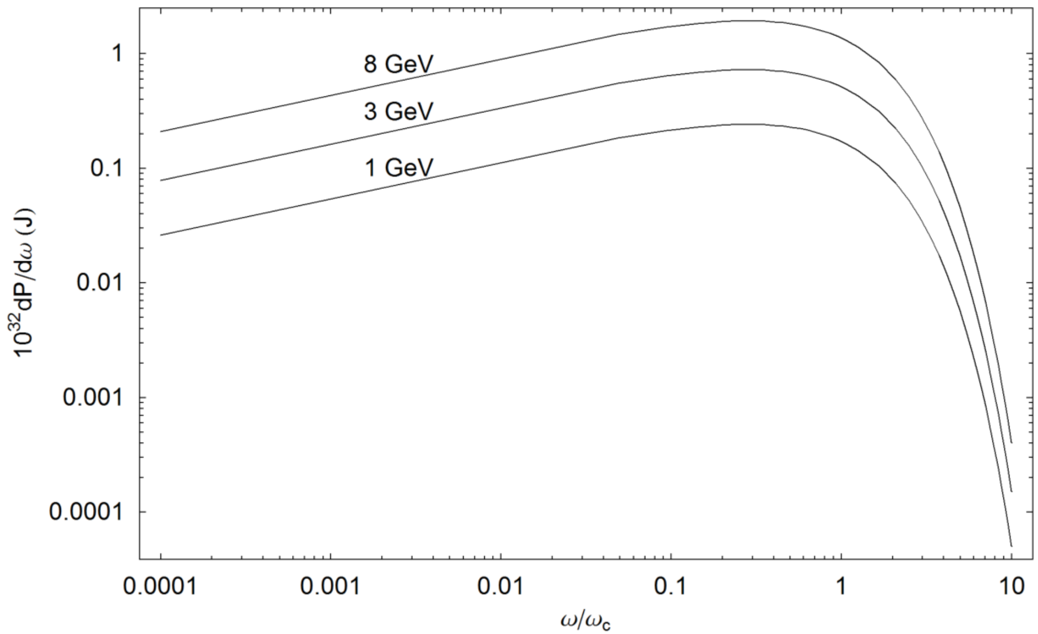
where q is the electric charge of the particle, m is the (rest) mass of the particle, and ϵ_0 is the permittivity of free space. For electrons, $|q| = e = 1.602 \times 10^{-19}$ C,

$$C_\gamma = 8.846 \times 10^{-5} \text{ m GeV}^{-3}. \tag{3}$$

The synchrotron radiation from a charged particle in a dipole magnet extends over a broad range of wavelengths: typically significant amounts of power in the part of the EM spectrum ranging from the infra-red up to the ultra-violet or soft x-ray regions.

The peak in the radiation spectrum occurs close to the critical frequency given by

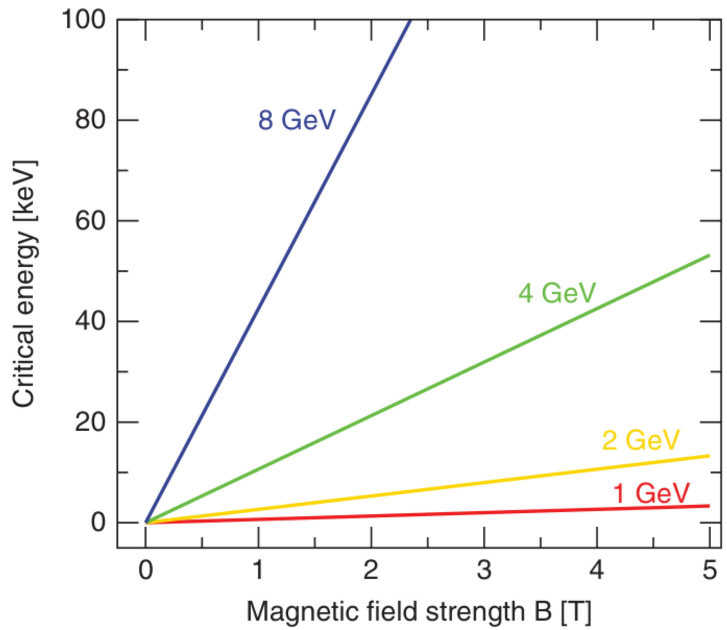
$$\omega_c = \frac{3c}{2\rho}\gamma^3. \tag{4}$$



Above ω_c, the radiation power fall rapidly with increasing frequency, and there is also a lower energy tail, as well as some fraction of higher energy photons.

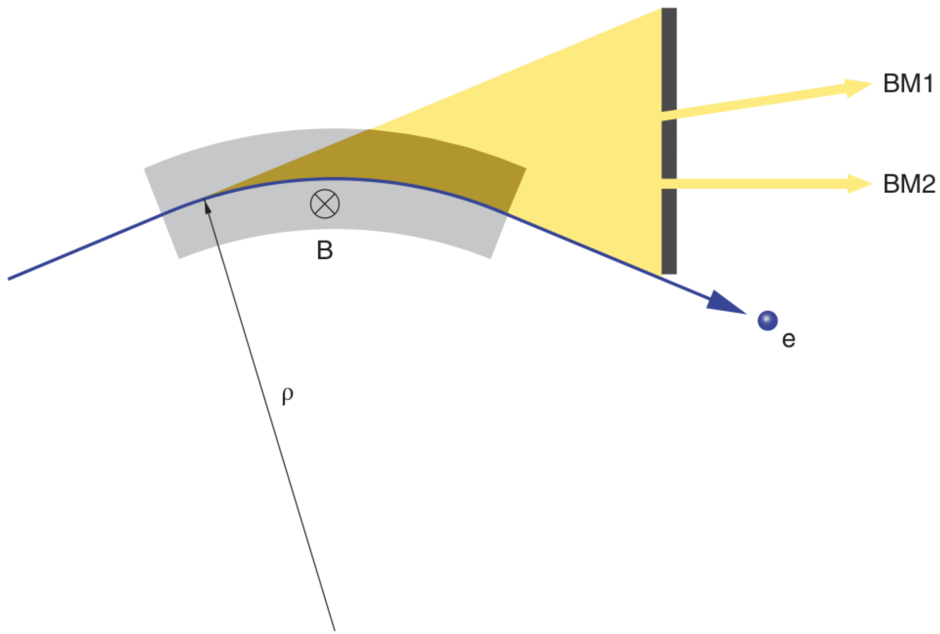
For efficient production of hard x-ray (above 5–10 keV, below 0.2–0.1 nm wavelength), which is important for a number of scientific fields, insertion devices (undulators and wigglers) can be used.

The critical energy is $E_c = \hbar\omega_c = \hbar\frac{2\pi c}{\lambda_c}$.



The synchrotron radiation from a single particle is emitted in a narrow cone around the instantaneous direction of motion of the particle. The opening angle of the radiation is roughly $1/\gamma$.

In a dipole magnet, particles emit radiation along the entire length of the magnet, and since the bending angle of a dipole in a storage ring is typically large compared to $1/\gamma$, the radiation takes the form of a 'fan' with narrow vertical divergence, but horizontal divergence roughly equal to the dipole bending angle (of order 10°).



The third-generation light sources are designed to optimise the radiation from insertion devices.

- Wigglers are designed so that the amplitude of the trajectory oscillation around the axis of the device is large. The radiation power spectrum from a wiggler is then similar to that from a dipole magnet, and the horizontal divergence is dominated by the change in angle of the beam trajectory along the device. (cf., damping wiggler)

- In an undulator, however, the amplitude of the trajectory oscillation around the axis is much smaller. This leads to interference effects between the radiation produced in successive periods of the array of magnets, and results in a radiation power spectrum dominated by a series of sharp spikes (essentially, a line spectrum). The angular divergence of undulator radiation is also dominated by the intrinsic opening angle of radiation from individual particles (of order $1/\gamma$) rather than by the trajectory of the beam.

A detailed analysis of radiation emission in an undulator leads to the undulator equation giving the wavelengths λ_n of the spectral lines,

$$\lambda_n = \frac{\lambda_u}{2n\gamma^2}\left(1 + \frac{K^2}{2} + \theta^2\gamma^2\right), \tag{5}$$

- λ_u is the undulator period.
- θ is the angle with respect to the axis of the undulator at which the radiation is observed. $\theta = 0$ corresponds to the forward direction.
- n is an integer corresponding to the different harmonics (lines) in the radiation spectrum. For $\theta = 0$, only odd harmonics are observed.
- K is deflection parameter that characterises the amplitude of the oscillation of the trajectory around the axis of the insertion device. ($K > 1$ for a wiggler, $K < 1$ for an undulator)

Although the radiation power P_γ is a significant quantity, a more important figure of merit is often the radiation beam brightness.

The brightness is defined as the amount of radiation in a given frequency range, per unit area in phase space.

The area of phase space occupied by electrons is quantified by the emittance, which is a constant as the beam moves around the ring.

The photon beam brightness is therefore independent of the location of the source of the radiation in the storage ring.

Since the radiation from each electron is emitted in a narrow cone around the instantaneous direction of motion of the electron, the brightness is also a measure of the power per unit phase space area, and is a constant as the radiation is transported from the storage ring to an experimental area.

The brightness (or brilliance) can be calculated from the formula:

$$\mathcal{B} = \frac{\Phi_\gamma}{4\pi^2 \Sigma_x \Sigma_{x'} \Sigma_y \Sigma_{y'} (d\omega/\omega)}. \tag{6}$$

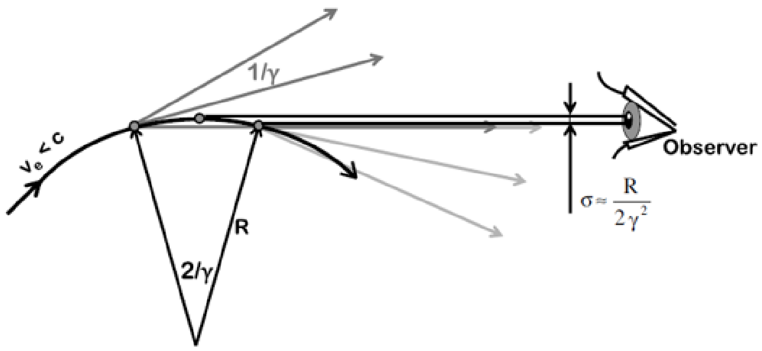
- Φ_γ is the photon flux (number of photons produced per second) at angular frequency ω.
- $d\omega/\omega$ is the bandwidth (typically 0.1%).
- $\Sigma_{x,y}$ and $\Sigma_{x',y'}$ are given by summing in quadrature the electron/photon beam sizes, and beam divergences.

$$\Sigma_{x,y} = \sqrt{\sigma_{x,y}^2 + \sigma_r^2}, \;\; \Sigma_{x',y'} = \sqrt{\sigma_{x',y'}^2 + \sigma_{r'}^2} \tag{7}$$

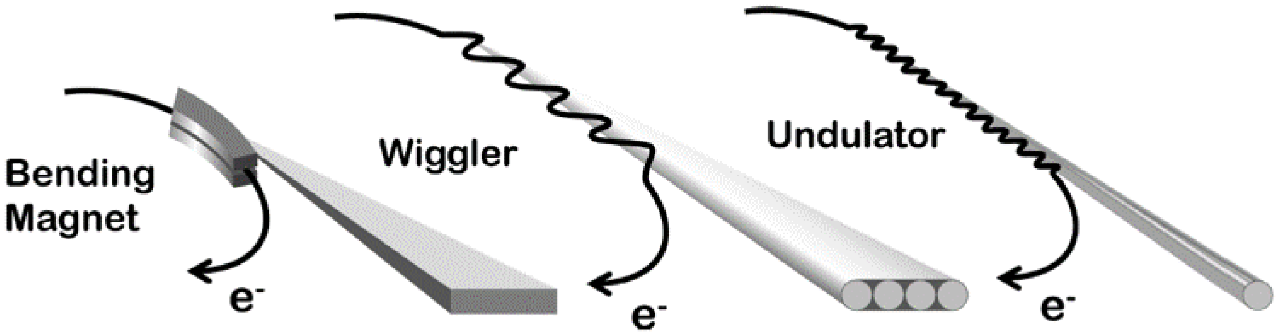
- The intrinsic photon-beam size and divergence by an undulator of length L and emitting photons at a wavelength λ:

$$\sigma_r = \frac{1}{2\pi}\sqrt{\frac{\lambda L}{2}}, \;\; \sigma_{r'} = \sqrt{\frac{\lambda}{2L}}, \quad \sigma_r \sigma_{r'} = \frac{\lambda}{4\pi} = \epsilon_r. \tag{8}$$

It is often helpful to have a beam with a small divergence for higher intensity of the radiation in experiments. For a single relativistic charged particle, divergence is $\sigma_{\gamma'} \sim 1/\gamma$.



In a third-generation synchrotron light source, a typical electron energy is 3 GeV, so γ is of order 6000; the synchrotron radiation can then have an opening angle of a fraction of a mrad, i.e. of order 0.01°.



Consider an ID that produces a magnetic field varying sinusoidally along the length of the device, with amplitude B_0 and period λ_u.

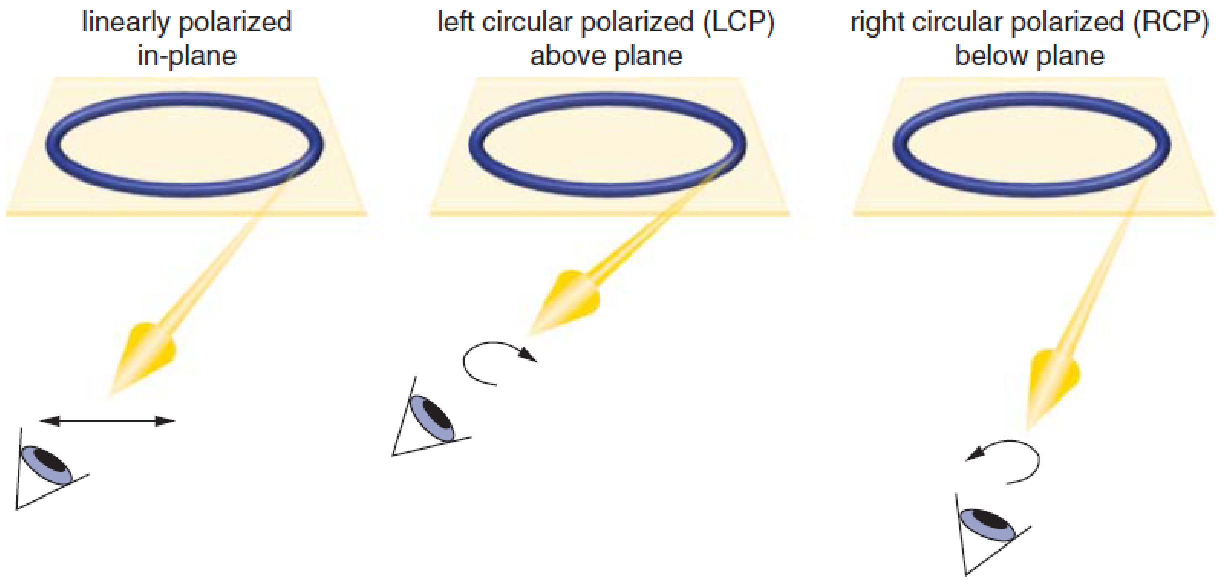
Electrons will follow a sinusoidal trajectory through the device with the peak angular deflection K/γ.

$$K = \frac{eB_0}{mc}\frac{\lambda_u}{2\pi} \approx 93.36\ B_0[\mathrm{T}]\ \lambda_u[\mathrm{m}] \tag{9}$$

- In an undulator, $K < 1$; the divergence of the radiation beam will then be dominated by the intrinsic opening angle of the radiation $(1/\gamma)$ from an individual particle.
- In a wiggler, however, $K > 1$ and the divergence of the radiation beam will be dominated by the angular deflection of particle trajectories through the wiggler.

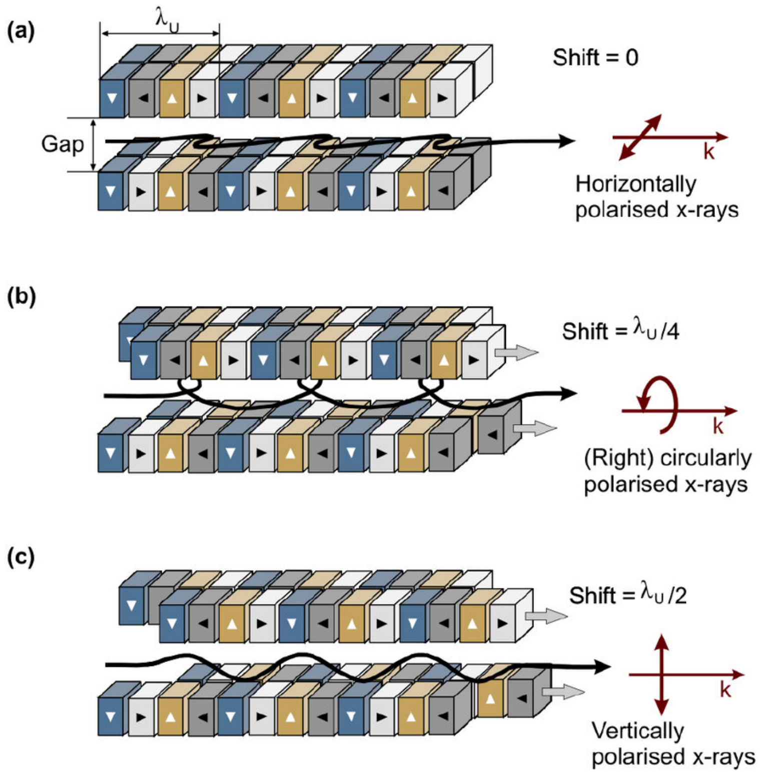
Synchrotron radiation from a dipole magnet bending a beam in the horizontal plane is horizontally polarised (i.e. the electric field oscillates in the horizontal plane) when observed in the horizontal plane.

Radiation emitted at (small) angles above or below the horizontal plane has a vertical component of polarisation in addition to the horizontal component.



IDs in which the field polarity alternates along the length of the device produce radiation with similar polarisation properties to dipoles.

More sophisticated IDs can produce elliptical polarisation, in addition to planar polarisation. This can be achieved by moving the magnetic poles relative to one another longitudinally [Advanced Planar Polarized Light Emitting (APPLE) type].



Synchrotron radiation leads to the horizontal, vertical, and longitudinal emittances of a beam in an electron storage ring reaching equilibrium values determined by the beam energy and the design of the magnetic lattice.

- The energy loss from synchrotron radiation leads to an exponential decrease (damping) of the amplitudes of synchrotron and betatron oscillations of any electron in a storage ring.
- However, the quantum nature of the radiation prevents the amplitudes damping to zero: the random emission of photons results in some excitation of synchrotron and betatron oscillations.

The equilibrium emittances are determined by the balance between radiation damping and quantum excitation.

The combined effects of the RF cavities in a storage ring and phase slip (from dispersion in the dipole magnets) led to particles performing synchrotron oscillations. Let us write down the equations of motion for the longitudinal coordinate z and the energy deviation $\delta = \Delta E/E_0$ of an electron in a storage ring.

[Energy] On each turn, the electron acquires energy from the RF cavities ($V_{\mathrm{RF}}, \omega_{\mathrm{RF}}$), and loses energy through synchrotron radiation. Averaging the energy gain and loss over a single turn, the rate of change of the energy deviation can be written

$$\frac{d\delta}{dt} = \frac{qV_{\mathrm{RF}}}{E_0 T_0} \sin\left(\phi_s - \frac{\omega_{\mathrm{RF}} z}{c}\right) - \frac{U}{E_0 T_0}, \qquad (10)$$

where E_0 is the reference energy, T_0 is the revolution period, and U is the total energy lost through synchrotron radiation.

The synchronous phase ϕ_s is defined so that the energy gained by the reference particle ($z = 0$) in an RF cavity is exactly matched by the energy U lost over one turn (hence $\delta = 0$).

In other words,

$$qV_{\mathsf{RF}}\sin(\phi_s) = U_0, \ \ U \approx U_0 + E_0\delta \left.\frac{dU}{dE}\right|_{E=E_0}. \tag{11}$$

[Coordinate] The rate of change of the longitudinal co-ordinate is related to the energy deviation by the phase slip factor η_p. Again averaging over a single turn, the rate of change of the longitudinal co-ordinate is

$$\frac{dz}{dt} = -\eta_p c\delta. \tag{12}$$

If we assume that z is small, so that $\omega_{\mathsf{RF}}z/c \ll 1$, we can make the approximation

$$\sin\left(\phi_s - \frac{\omega_{\mathsf{RF}}z}{c}\right) \approx \sin(\phi_s) - \cos(\phi_s)\frac{\omega_{\mathsf{RF}}z}{c}. \tag{13}$$

The equations of motion are then linear in the variables z and δ, and can be combined to give the following equation for the energy deviation:

$$\frac{d^2\delta}{dt^2} + 2\alpha_{\mathrm{E}}\frac{d\delta}{dt} + \omega_s^2\delta = 0. \tag{14}$$

The synchrotron frequency is $\omega_s = 2\pi\nu_s/T_0$, with ν_s is the synchrotron tune (the number of synchrotron oscillations completed per turn of the ring). The constant α_{E} is given by the change of energy lost through synchrotron radiation with respect to a change in the energy of the particle, evaluated at the reference energy:

$$\alpha_{\mathrm{E}} = \frac{1}{2T_0}\frac{dU}{dE}\bigg|_{E=E_0}. \tag{15}$$

The equation of motion for the energy deviation is the equation for a damped harmonic oscillator, with angular frequency ω_s and damping time $\tau_z = 1/\alpha_{\mathrm{E}}$.

The solution to the equation of motion can be written

$$\delta(t) = \delta_0 e^{-\alpha_\mathrm{E} t} \sin(\omega_s t + \theta_0), \tag{16}$$

where δ_0 is the initial amplitude, and θ_0 is the initial phase of the oscillation.

We find that the longitudinal coordinate z obeys a similar equation of motion:

$$z(t) = \frac{\eta_p c}{\omega_s} \delta_0 e^{-\alpha_\mathrm{E} t} \cos(\omega_s t + \theta_0), \tag{17}$$

The main significance of these results is that synchrotron radiation leads to damping (exponential decay of the amplitude) of synchrotron oscillations.

To complete the description of the motion, however, we need to find an expression for the damping constant α_E in terms of the parameters of the storage ring and the electron beam.

The energy loss per turn is found by integrating the radiation power over a complete turn of the ring, taking into account the dependence of the revolution period on the energy deviation (because of the presence of dispersion η_x).

First, calculate the energy loss per turn for a (ultra-relativistic) particle with zero energy deviation:

$$U_0 = \oint P_\gamma(E_0)dt = \oint P_\gamma(E_0)\frac{ds}{c} = \frac{C_\gamma}{2\pi}E_0^4 \oint \frac{1}{\rho^2}ds = \frac{C_\gamma}{2\pi}E_0^4 I_2, \qquad (18)$$

where I_2 is the second synchrotron radiation integral. (we may eliminate C_γ in terms of I_2)

In an isomagnetic ring, where the radius of curvature of the reference trajectory is constant around the entire circumference,

$$I_2 = \oint \frac{1}{\rho^2}ds = \frac{2\pi}{\rho}. \qquad (19)$$

Second, calculate the energy loss per turn for a (ultra-relativistic) particle with zero betatron amplitude but with an offset $x = \eta_x \delta$:

$$U = \oint P_\gamma(E) dt = \oint P_\gamma(E) \frac{dC}{c} = \oint P_\gamma(E) \left(1 + \frac{\eta_x}{\rho}\delta\right) \frac{ds}{c}. \tag{20}$$

We find (after some algebra)

$$\left.\frac{dU}{dE}\right|_{E=E_0} = j_z \frac{U_0}{E_0}, \tag{21}$$

where j_z is the longitudinal damping partition number:

$$j_z = 2 + \frac{I_4}{I_2}. \tag{22}$$

I_4 is the fourth synchrotron radiation integral:

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1\right) ds, \tag{23}$$

where $k_1 = (q/P_0)(\partial B_y/\partial x)$ is the quadrupole gradient in the dipole field (i.e., where ρ is finite). It should be mentioned that $\rho = |L|/\theta$ can have opposite signs when the bending is in the reversed direction.

Finally, we can write the longitudinal damping time:

$$\tau_z = \frac{1}{\alpha_\mathrm{E}} = \frac{1}{\frac{1}{2T_0}\frac{dU}{dE}\big|_{E=E_0}} = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{24}$$

Note that the focusing strength only affects the value of I_4 if the quadrupole field component appears in a dipole magnet (e.g., combined function, edge focusing):

$$B = B_0 + x\frac{\partial B_y}{\partial x}. \tag{25}$$

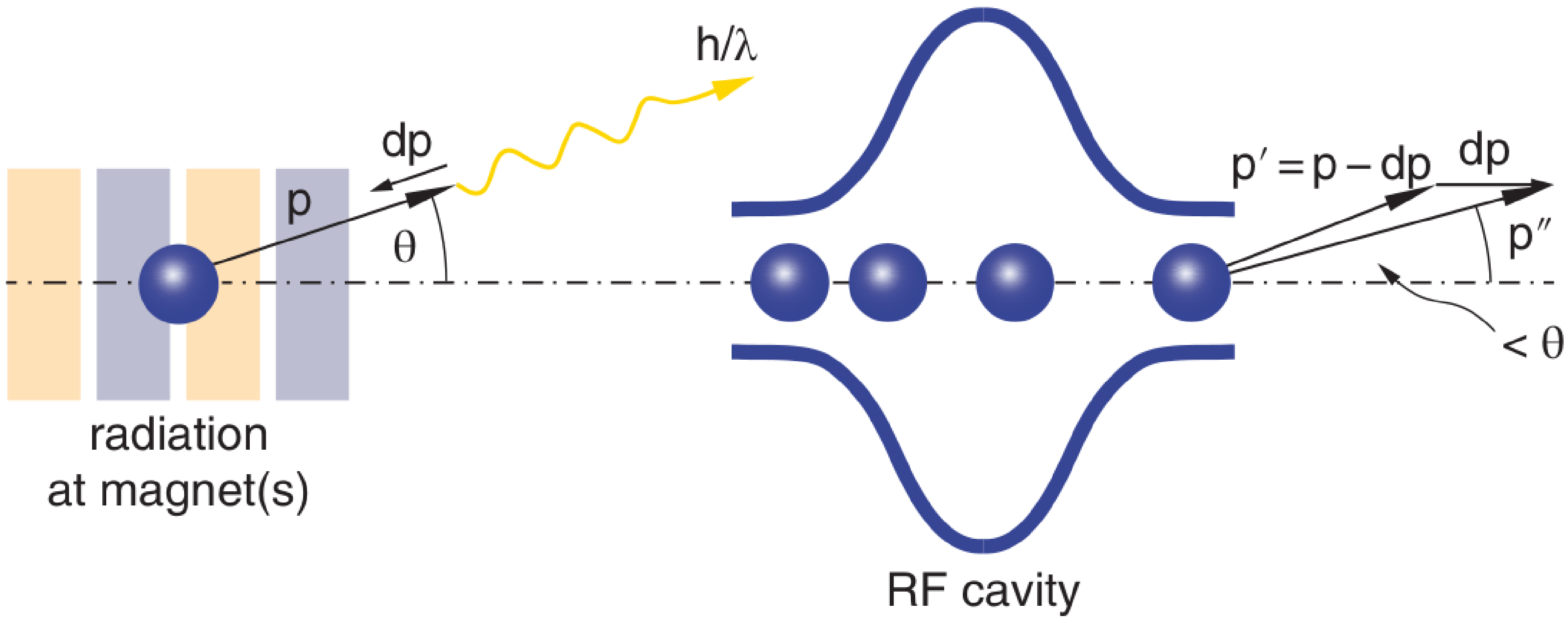
If the dipole magnets in a storage ring have no quadrupole component (i.e. the field is uniform, and does not vary with x or y), then $I_4 \ll I_2$, and $j_z \approx 2$.

In our analysis of the effects of synchrotron radiation on synchrotron motion in a storage ring, we were able to average the radiation energy loss over a complete revolution of a particle around the ring: this was a valid approach, because in most cases the synchrotron tune is small, $\nu_s \ll 1$, so that the changes in the co-ordinate z and energy deviation δ are small over one turn.

In the case of betatron oscillations, however, particles will usually complete many oscillations over a single turn of the ring, so we cannot assume that we can average the effects of synchrotron radiation over one turn.

Suppose that a particle emits some synchrotron radiation as it passes through a dipole magnet. The energy and momentum of the particle will fall as a result.

Since the radiation is emitted (for ultrarelativistic particles) along the instantaneous direction of motion of the particle, the emission of synchrotron radiation will not change the direction in which the particle is moving (unless RF cavity is used).



If the change in momentum is $\Delta P > 0$ and the particle initially has a momentum close to the reference momentum P_0, then each component of the momentum vector will scale by a factor (approximately) $1 - \Delta P/P_0$.

$$P - \Delta P \approx P\left(1 - \frac{\Delta P}{P_0}\right). \tag{26}$$

In particular, the vertical momentum will undergo a change $\Delta p_y = p_y\left(-\frac{\Delta P}{P_0}\right)$, which leads to a change in the vertical betatron action J_y.

$$J_y = \frac{1}{2}\left(\gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2\right), \;\; \Delta J_y = -\left(\alpha_y y p_y + \beta_y p_y^2\right)\frac{\Delta P}{P_0}. \tag{27}$$

Averaging over all particles in the beam (assumed uniformly distributed in betatron phase angle) we find

$$\langle \Delta J_y \rangle = -\varepsilon_y \frac{\Delta P}{P_0}, \tag{28}$$

where $\varepsilon_y = \langle J_y \rangle$ is the vertical emittance of the beam. (see p. 2-20)

If the total change in betatron action over a single turn of the ring is small, we can find the rate of change of the betatron action by averaging the changes over one revolution: (though the changes in y and p_y can be large due to many betatron oscillations)

$$\frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0}\oint \frac{\Delta P}{P_0} \approx -\frac{\varepsilon_y}{T_0}\oint \frac{\Delta E}{E_0} = -\frac{U_0}{E_0 T_0}\varepsilon_y = -\frac{2}{\tau_y}\varepsilon_y. \tag{29}$$

Here, we define the vertical damping time

$$\tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0 = 2\frac{E_0}{U_0}T_0. \tag{30}$$

The vertical emittance falls exponentially, with damping time $\tau_y/2$:

$$\varepsilon_y(t) = \varepsilon_y(0)e^{-2t/\tau_y}. \tag{31}$$

The factor 2 in the definition of the damping time is a matter of convention, $j_y = 1$ is introduced for formal completeness.

The energy lost by particles through synchrotron radiation is replaced in electron storage rings by RF cavities.

Since RF cavities are generally designed so that the accelerating field is parallel to the reference trajectory, although the momentum of a particle may increase (back to the original reference value) as it passes through the cavity, only the longitudinal component will change, and the transverse components will remain the same.

The horizontal and vertical betatron actions of a particle passing through an RF cavity will therefore be unchanged by the fields in the cavity, and the cavity will have no effect on the horizontal or vertical emittance.

[Note] If the reference momentum P_0 is increased by RF as in linac, the scaled vertical momentum of a particle, $p_y = P_y/P_0$, is decreased (adiabatic damping). In a storage ring, if the beam is held at a fixed energy, there is no adiabatic damping.

Calculating the horizontal damping time is more complicated than calculating the vertical damping time, for three reasons.

1. As a particle emits EM radiation, there will be a (classical or continuous) change in the closed orbit about which the trajectory of the particle oscillates (effects of dispersion). This results in a change in the betatron amplitude, in addition to the change resulting directly from the change in horizontal momentum of the particle.

2. Where the reference trajectory is curved, the path length taken by a particle depends on the horizontal co-ordinate.

3. Dipole magnets are sometimes constructed with a quadrupole field component, so the field strength seen by a particle in a dipole magnet depends on the horizontal co-ordinate of the particle.

Taking the relevant effects into account, we nonetheless find that the horizontal emittance damps exponentially, in the same way as the longitudinal and vertical emittances:

$$\varepsilon_x(t) = \varepsilon_x(0)e^{-2t/\tau_x}. \tag{32}$$

The horizontal damping time τ_x, however, has to take account of 1) the effects of dispersion, 2) the curvature of the reference trajectory, and 3) the possible presence of a quadrupole field component in the dipole magnets:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \tag{33}$$

where j_x is the horizontal damping partition number

$$j_x = 1 - \frac{I_4}{I_2}. \tag{34}$$

If the dipole magnets have no quadrupole field component, then $I_4 \ll I_2$ and hence $j_x \approx 1$.

Inclusion of a quadrupole field component in the dipole magnets (through I_4), however, makes it possible to 'shift' the radiation damping between the longitudinal and horizontal motion.

For a planar storage ring with (without vertical dispersion),

$$j_x + j_y + j_z = 4, \ \frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_z} = \text{const.} \tag{35}$$

The sum of the damping partition number is independent of the lattice design (i.e., I_2, I_4).

The Robinson damping theorem states that although it is possible to shift the balance of the radiation damping provided between the different degrees of freedom, the overall rate of damping is fixed by the beam energy (E_0) and the rate of energy loss (U_0/T_0).

[1] Introduction to Beam Dynamics in High-Energy Electron Storage Rings, Andrzej Wolski.

[2] Beam Dynamics In High Energy Particle Accelerators, Andrzej Wolski.

[3] An Introduction to Synchrotron Radiation: Techniques and Applications, Philip Willmott.

[4] Unifying Physics of Accelerators, Lasers and Plasma, Andrei Seryi.