4. Many Particle Dynamics

In this Lecture we introduce a method to analyze particle dynamics in circular accelerators. The method is based on the theory developed by Courant and Snyder in the 1950s and has been popularized in accelerator community.

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Many particles

To study behavior of a particle beam (i.e. collection of particles) under the influence of EM fields, we may follow each particle's trajectory. However, a beam usually consists of many particles (e.g. 10^9 - 10^{10}), each with different initial conditions. It does not make sense to follow each particle, which is too cumbersome or impossible even with a computer. We therefore need to devise some methods.

One such method is to observe a beam in phase space.

- Each particle is represented by a single point in 6-dimensional phase space (x, p_x, y, p_y, z, p_z).
- In beam physics, except some special cases, we can treat the motion independently in different degree of freedom.
- The location of a particle in phase space can then be defined by 3 pairs of conjugate variables, $(x, p_x), (y, p_y), (z, p_z)$.
- At least in existing accelerators, coupling of motions between different degrees of freedom is usually small and can be treated as a perturbation.

To describe particle beams in phase space we use two conjugate variables such as

$$(x, p_x), \qquad (y, p_y), \qquad (z, p_z)$$

Instead of (z, p_z) it is customary to use

$(-t, E)$ or (ϕ, E) or $(-\Delta t, \Delta E)$ or $\left(-t, \dfrac{\Delta E}{E_0}\right)$ or $\left(\phi, \dfrac{\Delta E}{E_0}\right)$

or $\left(s - vt, \delta = \frac{p - p_0}{p_0}\right)$ or $\left(\frac{s}{\beta_0} - ct = -c(t - t_0),\ p_t = \frac{E}{cp_0} - \frac{1}{\beta_0}\right)$ etc.

where $\Delta t = t - t_0(s),\ \Delta E = E - E_0$

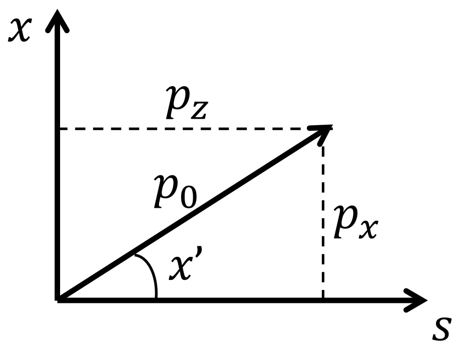
Longitudinal oscillation is much smaller (slower) than the transverse oscillation and therefore it is usually treated separately.

(x, x'), (y, y') phase spaces

Use of (x, p_x), (y, p_y) is not convenient because the momenta are very small numbers in general.

$$m\frac{dx}{dt} = p_x \rightarrow m\frac{dx}{ds/v} = p_x \rightarrow \frac{dx}{ds} \equiv x' \approx \frac{p_x}{p_0} \qquad p_0 = mv$$

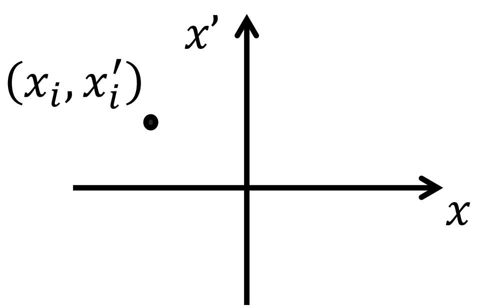
We use paraxial beam optics meaning $x' \ll 1$ and $p_x \ll p_0$. Then $p_x = p_0 x' \approx p_0 \sin x'$.



$$\sin x' = \frac{p_x}{p_0}$$

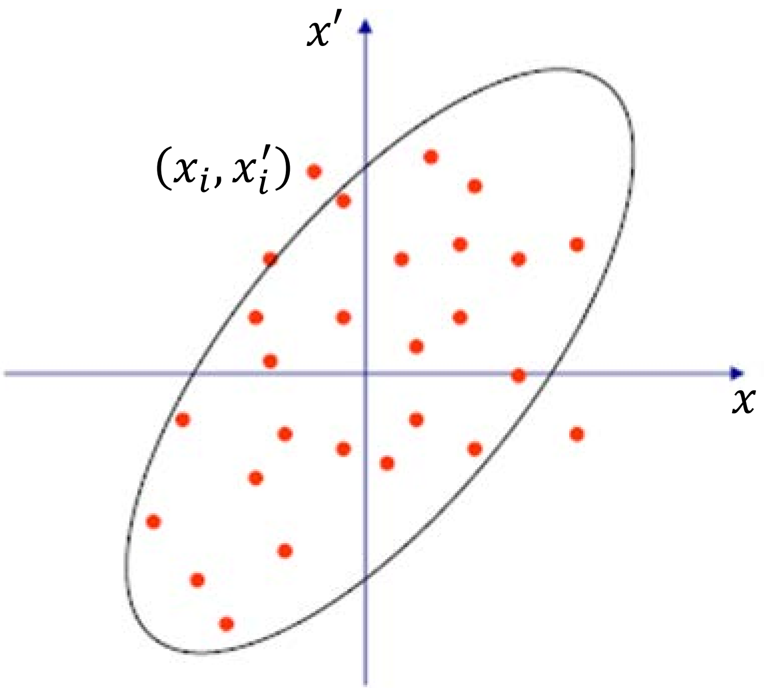
Momentum $p_x \approx p_0 \sin x'$

For beamlines with constant momentum/energy it is customary to use (x, x'), (y, y').



Note that (x, x') is not a set of canonically conjugate variables, but to a good (paraxial) approximation they can be considered as a set of phase-space coordinates.

Many particles



Randomly distributed particles can be enclosed by an ellipse, the phase-space ellipse.

Phase-space area of a beam

To describe particle dynamics in phase space, we begin with some analytical geometry dealing with properties of an ellipse. Later we shall see that an ellipse well suits to describe particle beams.

General equation for an ellipse can be written in the form of a bi-linear product

$$X^T \Sigma^{-1} X = 1 \qquad (1) \qquad\qquad X^T = (x, x', y, y', z, p_t, \cdots)$$

where X is a column matrix (or vector) consisting of phase-space coordinates (and others such as the spin) and Σ is a matrix describing the ellipse, which we call the “beam (sigma) matrix” or “beam-envelope matrix” as will be obvious soon.

Volume of n-dimensional ellipsoid is given by (from analytic geometry)

$$V_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)} \sqrt{\det(\Sigma)} \qquad (2)$$

s : reference particle’s long. position

where Γ is the gamma function.

The beam (sigma) matrix is not to be confused with the transfer matrix or map that we have considered before.

Beam (sigma) matrix

To make our discussion easy and transparent, let's consider two-dimensional case.

In two-dimensional (phase) space, $n = 2$, and therefore with $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$

$$V_2 = \pi\sqrt{\det(\Sigma)} = \pi\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \equiv \pi\epsilon \qquad (\sigma_{12} = \sigma_{21}) \qquad (3) \qquad \sqrt{\det(\Sigma)} = \epsilon$$

where $\pi\epsilon$ is the area of the phase-space ellipse. Here ϵ is called the beam emittance.

Thus the beam emittance is the phase-space area occupied by a beam divided by π.

From $X^T\Sigma^{-1}X = 1$ we get with $\Sigma^{-1} = \frac{1}{\det(\Sigma)}\begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$ (4) $\qquad X^T = (x, x')$

the equation of ellipse

$$\sigma_{22}x^2 - 2\sigma_{12}\, xx' + \sigma_{11}x'^2 = \det(\Sigma) = \epsilon^2 \qquad (5)$$

(Again note that $\sigma_{21} = \sigma_{12}$)

In terms of ellipse parameters (see the Figure in the next page) the beam (sigma) matrix is defined in the form:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \equiv \epsilon\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \qquad (6)$$

$$\Sigma^{-1} = \frac{1}{\epsilon^2}\begin{pmatrix} \epsilon\gamma & \epsilon\alpha \\ \epsilon\alpha & \epsilon\beta \end{pmatrix} = \frac{1}{\epsilon}\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$$

Then Eq. (5) becomes the well-known Courant-Snyder invariant:

$$\gamma x^2 + 2\alpha xx' + \beta x'^2 = \epsilon \qquad (7)$$

Propagation of the beam (sigma) matrix

To find the propagation of the beam matrix along the beam transport line, let's begin with the equation of ellipse given by Eq. (1) at a longitudinal position z_1:

$$X_1^T \Sigma_1^{-1} X_1 = 1$$

At z_2 it is written as

$$X_2^T \Sigma_2^{-1} X_2 = 1$$

If we know the transfer matrix M between z_1 and z_2, then $X_2 = MX_1$ or $X_1 = M^{-1}X_2$.

From these, we can relate Σ_1 and Σ_2

$$1 = X_2^T \Sigma_2^{-1} X_2 = X_1^T M^T \Sigma_2^{-1} M X_1 = X_1^T \Sigma_1^{-1} X_1$$

Then we find the propagation of the beam matrix at two different places:

$$\Sigma_1^{-1} = M^T \Sigma_2^{-1} M \quad \text{or} \quad \Sigma_2 = M \Sigma_1 M^T \qquad (8)$$

Thus if we know the beam matrix at some position z_1 and if we know the transfer matrix between z_1 and z_2, then we can find the beam matrix at z_2 using Eq. (8).

We note by taking the determinants on both sides of Eq. (8)

$$\det(\Sigma_2) = \det(\Sigma_1)$$

which indicates the preservation of the phase-space area (or volume).

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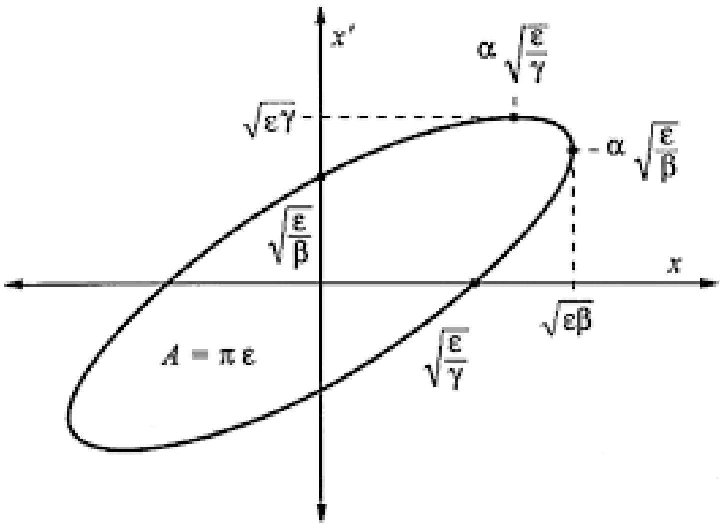
Phase space ellipse

The parameters α, β, γ in Eq. (6) are defined by an ellipse as



Note: These β, γ are different from the usual β, γ in relativity. We shall introduce them in more formal way soon.

We need beam dynamics studies to get some insight of these parameters. This is the subject of the Courant-Snyder theory.

β, α, γ are called Twiss parameters or Courant-Snyder parameters in the literature, and Eq. (7) is called as the Courant-Snyder invariant.

So beam emittance is an invariant quantity if the beam does not accelerate (or decelerate).

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Courant-Snyder parameterization: Betatron functions

In general, linearized equation of motion in circular accelerator for an on-momentum particle can be written as

$$\frac{d^2x(s)}{ds^2} + k(s)x(s) = 0 \qquad (9)$$

Periodic condition: $k(s+C) = k(s)$ $\qquad$ C: circumference or cell length

An equation of this form is called the Hill's equation; a linear second-order differential equation with periodic boundary conditions.

If $k\ =$ constant, the solution would be $x(s) = c_1\cos\left(\sqrt{k}s + \phi\right)$, with c_1 and ϕ being constant. But in Eq. (9) k is a function of s. So we try "variation of parameters" method.

Let's assume a solution for the i^{th} particle of the form

$$x_i(s) = \underbrace{a_i}_{\text{const}} \sqrt{\beta(s)}\cos[\psi(s) + \underbrace{\phi_i}_{\text{const}}] \qquad (10) \qquad \text{c.f.)}\ x(s) = \sqrt{2\beta_x(s)J_x}\cos\phi_x(s)$$

Inserting this into the Hill's eq. (9) and setting coefficients of sine and cosine terms to be separately zero, we get (omitting details)

1) Equation for betatron function

$$\frac{1}{2}\beta\frac{d^2\beta}{ds^2}-\frac{1}{4}\left(\frac{d\beta}{ds}\right)^2+\beta^2 k=1 \qquad (11)$$

or equivalently

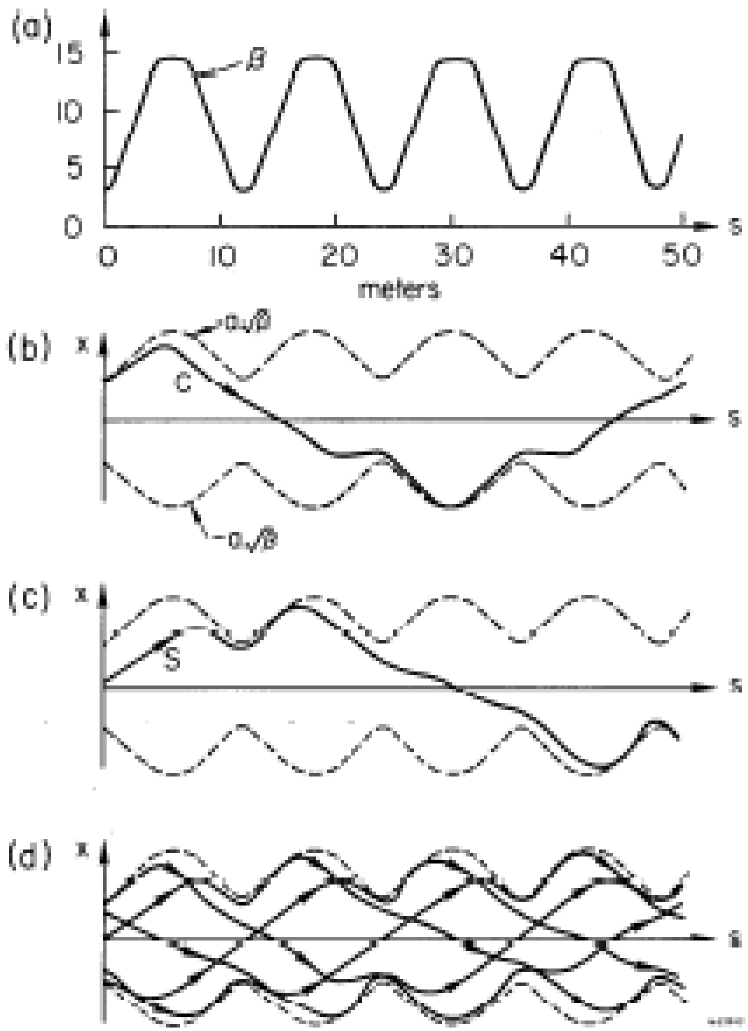
$$\frac{d^2}{ds^2}\sqrt{\beta(s)}+k(s)\sqrt{\beta(s)}-\beta(s)^{-3/2}=0 \qquad (12)$$

2) Betatron phase

$$\psi'(s)=\frac{d\psi}{ds}=\frac{1}{\beta(s)}\rightarrow\psi(s)=\int_0^s\frac{1}{\beta(\varsigma)}d\varsigma \qquad (13)$$

- Two different betatron functions are defined: one for horizontal (x) and the other for vertical (y) plane.
- For particular set of initial values of the betatron function, there is only one solution per plane for betatron function.
- For circular lattices, the initial value is equal to the final value at the end of one turn or one cell (i.e. periodic condition).
- The value of betatron function by definition is real and always positive.
- The unit of betatron function is meter.
- The constants multiplied by the square root of the betatron function [e.g. a_i in Eq. (10)] can be positive or negative to represent a particle's trajectory.

Betatron oscillations



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Let's go back to Eq. (10)

$$x_i(s) = a_i\sqrt{\beta(s)}\cos[\psi(s) + \phi_i] \qquad (10)$$

Taking derivatives with respect to s we get

$$x_i'(s) = a_i\frac{\beta'}{2\sqrt{\beta}}\cos[\psi(s) + \phi_i] - a_i\sqrt{\beta}\,\psi'\sin[\psi(s) + \phi_i]$$

Let $\alpha(s) = -\frac{1}{2}\beta'(s)$ and from $\psi'(s) = \frac{1}{\beta(s)}$ we write

$$x_i'(s) = -a_i\frac{1}{\sqrt{\beta}}[\alpha(s)\cos(\psi(s) + \phi_i) - \sin[\psi(s) + \phi_i]] \qquad (14)$$

Eliminating the phase $[\psi(s) + \phi_i]$ from Eqs. (10) and (14) and defining $\gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)}$ we get a similar from as Eq. (7):

$$\beta(s)x_i'^2 + 2\alpha(s)x_i x_i' + \gamma(s)x_i^2 = a_i^2 \qquad (15)$$

eq. of ellipse
called Courant-Snyder invariant

During betatron oscillation, each particle moves on its own ellipse with area $A_i = \pi a_i^2$ in phase space. Particle with maximum amplitude $\hat{a}$ defines whole beam with beam emittance $\varepsilon_x = \hat{a}^2$

Motion in phase space

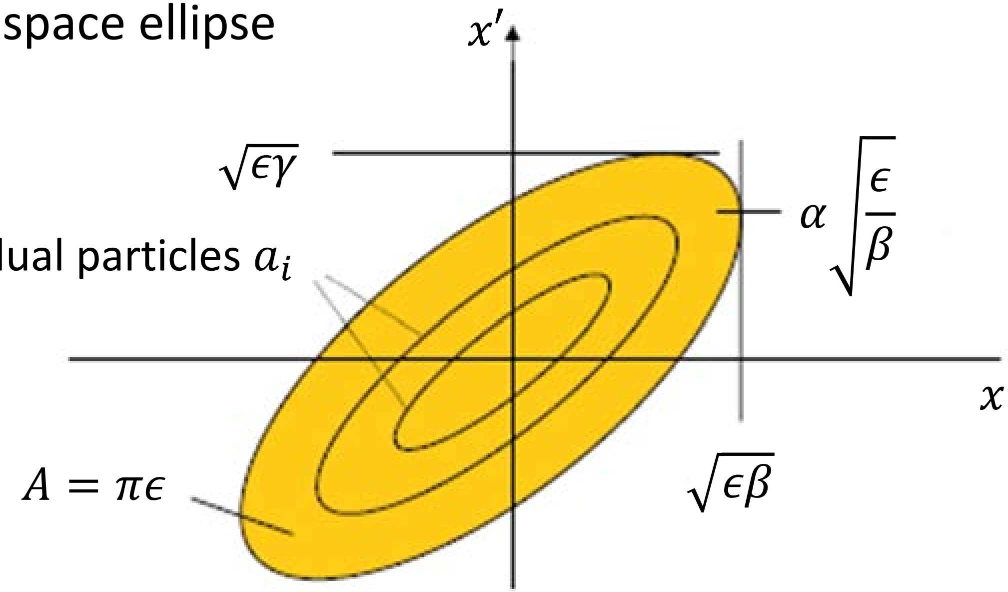
For bell shaped or Gaussian distribution of particles, we define the beam emittance by

$$\epsilon_u = \frac{\langle u^2 \rangle}{\beta_u} = \frac{1}{2}\langle a_i^2 \rangle \qquad u = x \text{ or } y \qquad (16)$$

$$\langle a_i^2 \rangle = \frac{1}{N}\sum_{i=1}^{N} a_i^2$$

N : number of particles

phase space ellipse



β, α, γ are also called Twiss parameters.

Beam envelope

Expression for single particle trajectory

$$x_i(s) = a_i\sqrt{\beta(s)}\cos[\psi(s) + \phi_i] \qquad (17)$$

can be used to define the beam envelope.

Look for all particles with maximum amplitude $\hat{a}_i$ and choose those with a phase such that $\cos[\psi(s) + \phi_i] = 1$ and find the beam envelope

$$R_{x,y}(s) = \pm\hat{a}_{x,y}\sqrt{\beta(s)} \qquad (18)$$

Note: We have never made use of the fact that these are charged particles. So far and for the rest of the Lecture, everything is applicable to both 1) charged particles beams and 2) photon beams.

We can calculate the particle trajectory

$$x_i(s) = a_i\sqrt{\beta(s)}\cos[\psi(z) + \phi_i]$$

for given initial conditions if we know the betatron functions.

Then how do we get the betatron functions? Do we have to integrate Eq. (12) for the betatron function?

$$\frac{d^2}{dz^2}\sqrt{\beta(s)} + k(s)\sqrt{\beta(s)} - \beta(s)^{-3/2} = 0 \qquad (12)$$

We note from this differential equation that there is only one solution for $\beta(s)$ and $\beta(s) > 0$ always!

Being nonlinear equation, Eq. (12) is difficult to solve.

We have already obtained the solution to this; Eqs. (8) and (6). But the introduction to Eq. (6) was not obvious there.

Fortunately however, there are other ways to get the solution: We shall develop a matrix formalism, which we can use to transform the betatron functions from one place to another. This, in factm has been shown in Eq. (8). But here, we shall be explicitly formulating the propagation of the Twiss parameters.

How do we do that? The answer is to use the Courant-Snyder invariant.

Let's start with the invariant at $s = 0$:

$$\beta(0)x_0'^2 + 2\alpha(0)x_0x_0' + \gamma(0)x_0^2 = \epsilon \qquad (19) \qquad \beta(0) = \beta(s=0) \text{ etc.}$$

Since this is invariant, it does not change at $s \neq 0$:

$$\beta(s)x'^2 + 2\alpha(s)xx' + \gamma(s)x^2 = \epsilon$$

This allows us to derive relations between $s = 0$ and s. We use the phase-space transformation of a trajectory, i.e.

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \qquad \begin{aligned} x_0 &= x(s=0) \\ x_0' &= x'(s=0) \end{aligned}$$

to replace x_0 and x_0' in

$$\beta(0)x_0'^2 + 2\alpha(0)x_0x_0' + \gamma(0)x_0^2 = \beta(s)x'^2 + 2\alpha(s)xx' + \gamma(s)x^2 = \epsilon = \text{invariant}$$

$$\beta_0 = \beta(0)$$

With $\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$ where $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ (Appendix A in Lecture 2)

we get

$$\beta_0(-cx + ax')^2 + 2\alpha_0(dx - bx')(-cx + ax') + \gamma_0(dx - bx')^2$$
$$= \beta(s)x'^2 + 2\alpha(s)xx' + \gamma(s)x^2$$

After sorting we find

$$(a^2\beta_0 - 2ab\alpha_0 + b^2\gamma_0)x'^2 + 2[-ca\beta_0 + (ad+bc)\alpha_0 - bd\gamma_0]xx'$$
$$+(c^2\beta_0 - 2cd\alpha_0 + d^2\gamma_0)x^2 = \beta x'^2 + 2\alpha xx' + \gamma x^2$$

Comparing the coefficients, we get the new betatron functions as

$$\beta(s) = a^2\beta_0 - 2ab\alpha_0 + b^2\gamma_0$$

$$\alpha(s) = -ca\beta_0 + (ad+bc)\alpha_0 - bd\gamma_0$$

$$\gamma(s) = c^2\beta_0 - 2cd\alpha_0 + d^2\gamma_0$$

This can be written in matrix form

$$\begin{pmatrix}\beta(s)\\ \alpha(s)\\ \gamma(s)\end{pmatrix} = \begin{pmatrix} a^2 & -2ab & b^2 \\ -ca & ad+bc & -bd \\ c^2 & -2cd & d^2\end{pmatrix}\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix} \rightarrow \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS'+C'S & -SS' \\ C'^2 & -2C'S' & S'^2\end{pmatrix}\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix} \quad (20)$$

Thus based on single particle transformation matrices we can transform all betatron functions. Similar to this, we can obtain the propagation of the elements of the beam (sigma) matrix discussed in p. 8.

But how do we get the first set of betatron functions, i.e. $\beta_0, \alpha_0, \gamma_0$?

$$\gamma_0 = \frac{1+\alpha_0^2}{\beta_0}$$

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The answer is “by measurement”.

- We have to measure the size, shape and orientation of the phase space ellipse.
- But we can measure only the beam size, width and height.
- We need to “rotate” ellipse and measure the size.
- This can be done with quadrupoles upstream.
- We measure beam size and a function of quadrupole strength.

More details can be found in H. Wiedemann, Particle Accelerator Physics, 4^{th} ed. P.224.

Beam emittance

Let's go back to Eq. (10), $x_i(s) = a_i\sqrt{\beta(s)}\cos[\psi(s)+\phi_i]$

Taking a derivative with respect to s

$$\alpha = -\frac{\beta'}{2}$$

$$x_i'(s) = a_i\frac{\beta'}{2\sqrt{\beta}}\cos[\psi(s)+\phi_i] - a_i\sqrt{\beta(s)}\psi'\sin[\psi(s)+\phi_i]$$

$$\psi' = \frac{1}{\beta}$$

For N particles, we take averages of second-moments:

$$\langle x_i^2\rangle = \langle a_i^2\beta\cos^2(\psi+\phi_i)\rangle = \frac{1}{2}\langle a_i^2\rangle\beta = \epsilon\beta$$

$$\langle x_i'^2\rangle = \langle a_i^2\rangle\frac{\alpha^2}{\beta}\langle\cos^2(\psi+\phi_i)\rangle - \langle a_i^2\rangle\frac{\alpha}{\beta}\langle\cos(\psi+\phi_i)\sin(\psi+\phi_i)\rangle + \langle a_i^2\rangle\frac{1}{\beta}\langle\sin^2(\psi+\phi_i)\rangle$$

$$= \langle a_i^2\rangle\frac{1}{2}\frac{1+\alpha^2}{\beta} = \epsilon\gamma$$

$$\langle x_i x_i'\rangle = -\langle a_i^2\rangle\alpha\langle\cos^2(\psi+\phi_i)\rangle - \langle a_i^2\rangle\langle\cos(\psi+\phi_i)\sin(\psi+\phi_i)\rangle = -\epsilon\alpha$$

From these three relations we get the expression for the Rms (Root-mean-square) beam emittance for arbitrary particle distribution:

$$\epsilon = \sqrt{\langle x_i^2\rangle\langle x_i'^2\rangle - \langle x_i x_i'\rangle^2} \qquad (21) \qquad \Sigma = \langle XX^T\rangle = \left\langle\begin{pmatrix} x_i \\ x_i'\end{pmatrix}\begin{pmatrix} x_i & x_i'\end{pmatrix}\right\rangle = \begin{pmatrix}\langle x_i^2\rangle & \langle x_i x_i'\rangle \\ \langle x_i x_i'\rangle & \langle x_i'^2\rangle\end{pmatrix}$$

Eq. (6) is now obvious.

This expression is used to calculate the beam emittance for a collection of particles.

Particle trajectories and betatron functions

One can also write the transfer matrices in terms of the Twiss parameters.

Let's write the general solution of the Hill's equation in the following alternative (but equivalent) form:

$$x(s) = a\sqrt{\beta(s)}\,\cos\psi(s) + b\sqrt{\beta(s)}\,\sin\psi(s)$$

where a and b are constants to be determined by initial conditions.

Take a derivative with respect to s

$$x'(s) = -a\left\{\frac{\alpha(s)}{\sqrt{\beta(s)}}\cos\psi(z) + \frac{1}{\sqrt{\beta(s)}}\sin\psi(s)\right\} + b\left\{-\frac{\alpha(s)}{\sqrt{\beta(s)}}\sin\psi(s) + \frac{1}{\sqrt{\beta(s)}}\cos\psi(s)\right\}$$

Let's set at $s = 0$

$$x(s=0) = x_0, \quad x'(s=0) = x'_0, \quad \beta(s=0) = \beta_0, \quad \alpha(s=0) = \alpha_0, \quad \psi(s=0) = 0$$

With these, we can determine the two constants a and b

$$x_0 = a\sqrt{\beta_0} \rightarrow a = \frac{x_0}{\sqrt{\beta_0}}$$

$$x'_0 = -a\frac{\alpha_0}{\sqrt{\beta_0}} + b\frac{1}{\sqrt{\beta_0}} \rightarrow b = \frac{x_0}{\sqrt{\beta_0}}\alpha_0 + \sqrt{\beta_0}x'_0$$

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}[\cos\psi(s) + \alpha_0 \sin\psi(s)]x_0 + [\sqrt{\beta_0\beta(s)}\sin\psi(s)]x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_0\beta(s)}}[\{\alpha_0 - \alpha(s)\}\cos\psi(s) - \{1 + \alpha_0\alpha(s)\}\sin\psi(s)]x_0 + \sqrt{\frac{\beta_0}{\beta(z)}}\{\cos\psi(s) - \alpha(s)\sin\psi(s)\}x_0'$$

This can be expressed in matrix form

$$X(s) = M(s|s_0)X(s_0) \qquad \text{or} \qquad \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \qquad \begin{matrix} x_0 = x(s = s_0) \\ x_0' = x_0'(s = s_0) \end{matrix}$$

where

$$M(s|s_0) = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}(\cos\psi + \alpha_0\sin\psi) & \sqrt{\beta_0\beta(s)}\sin\psi \\ \frac{1}{\sqrt{\beta_0\beta(s)}}[(\alpha_0 - \alpha(s))\cos\psi - (1 + \alpha(s)\alpha_0)\sin\psi] & \sqrt{\frac{\beta_0}{\beta(s)}}(\cos\psi - \alpha_0\sin\psi) \end{pmatrix} \tag{22}$$

Eq. (22) can be written in another form:

$$\begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix} \tag{23}$$

Or if we define the betatron amplitude matrix

$$B(s) = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} \quad \text{and} \quad B^{-1}(s) = \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{pmatrix} \tag{24}$$

Then the transfer matrix in terms of Twiss parameters can be written as

$$\begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} = B(s) \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} B^{-1}(s_0) \tag{25}$$

From Eq. (22), we see that the transfer matrix for a periodic lattice is

$$M(C) = \begin{pmatrix} C_{unit} & S_{unit} \\ C'_{unit} & S'_{unit} \end{pmatrix} = \begin{pmatrix} \cos\Delta\psi + \alpha_0 \sin\Delta\psi & \beta_0 \sin\Delta\psi \\ -\gamma_0 \sin\Delta\psi & \cos\Delta\psi - \alpha_0 \sin\Delta\psi \end{pmatrix} \quad (26)$$

where $\beta_0 = \beta, \qquad \alpha_0 = \alpha, \qquad \gamma_0 = \gamma$

and $\Delta\psi$ is the phase advance per unit (cell): $\Delta\psi = \int_{unit} \frac{ds}{\beta}$

Note that $|\mathrm{Trace}[M(C)]| < 2$ for stable motion

An important parameter in synchrotrons and storage rings is the tune defined as

$$\nu = \frac{\Delta\psi(C)}{2\pi} = \frac{1}{2\pi}\oint \frac{ds}{\beta} \qquad (C = 2\pi R) \qquad (27)$$

where $\Delta\psi(C)$ is the phase advance per one revolution (i.e. N cell).

So the tune is the number of oscillaions per one revolution of the reference particle around the ring.

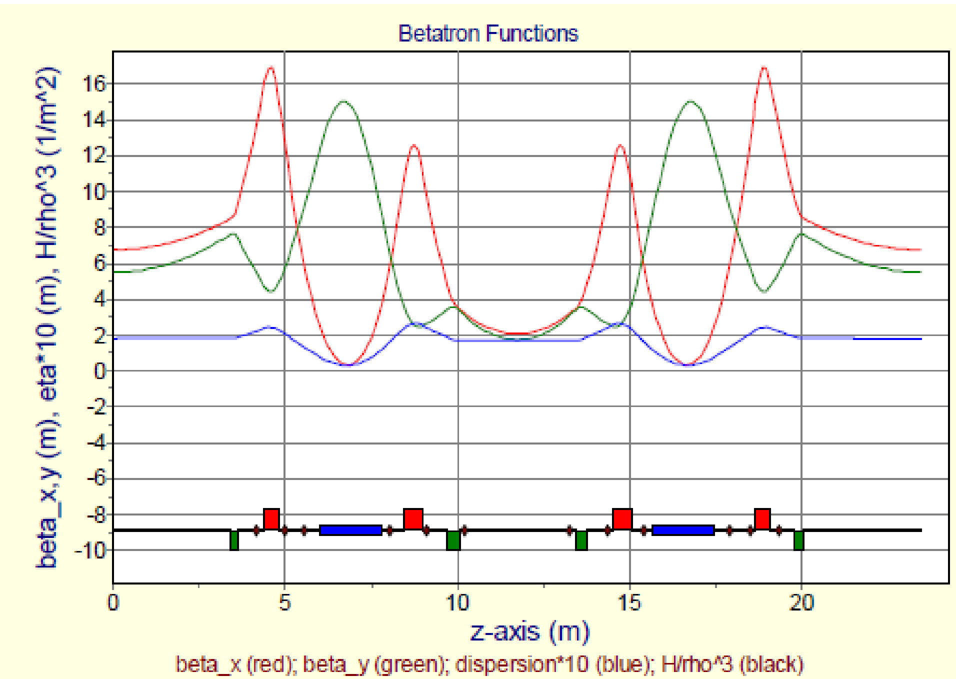
The larger the tune the stronger the focusing is.

Per each plane, the tune is defined; $\nu_x = 15.28, \nu_y = 9.18$ in PLS-2, $C = 281.82$ m

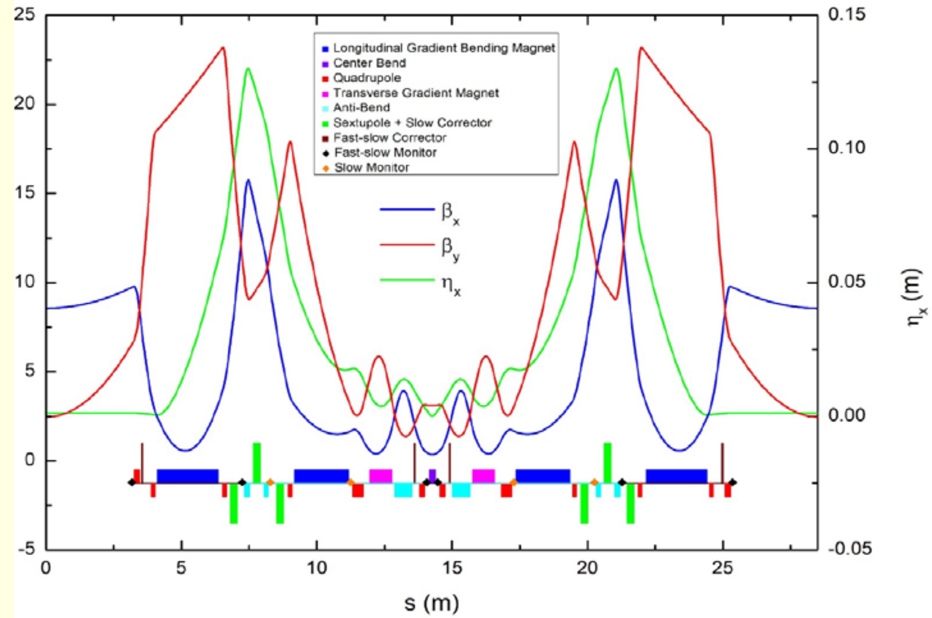
$\nu_x = 68.18, \nu_y = 23.26$ in 4GSR, $C = 799.297$ m

Tune may change with particle's oscillation amplitude, momentum deviation, etc.

Tunes must be carefully chosen to avoid possible resonances.



3.0 GeV PLS-2



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