2. Linear Maps I

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In Lecture 1 we have derived the scaled Hamiltonian in curved coordinates in canonical variables $\left(x,\frac{\pi_x}{p_0}\right)$, $\left(y,\frac{\pi_y}{p_0}\right)$ and $(-c\Delta t, p_t)$, which is given by

$$H=\frac{p_t}{\beta_0}-(1+hx)\sqrt{\left(\frac{1}{\beta_0}+p_t\right)^2-\left(\frac{\pi_x-qA_x}{p_0}\right)^2-\left(\frac{\pi_y-qA_y}{p_0}\right)^2-\frac{1}{\beta_0^2\gamma_0^2}}-(1+hx)\frac{qA_s}{p_0} \tag{1}$$

where

$$A_s=\boldsymbol{A}\cdot\boldsymbol{e}_s \quad \text{and} \quad h(s)=\frac{1}{\rho(s)} \tag{2}$$

Canonical (transverse) momenta are

$$\begin{aligned}\pi_x&=\gamma\beta_x mc+qA_x=p_x+qA_x\\ \pi_y&=\gamma\beta_y mc+qA_y=p_y+qA_y\end{aligned} \tag{3}$$

Longitudinal conjugate variables are

$$z=-c\Delta t=-c\big(t-t_0(s)\big)=\frac{s}{\beta_0}-ct,$$

$$p_t=\frac{E-E_0}{p_0c}=\frac{p_\tau}{c}=\frac{E}{p_0c}-\frac{1}{\beta_0}=\frac{1}{\beta_0}\frac{E-E_0}{E_0}=\frac{1}{\beta_0}\left(\frac{\gamma}{\gamma_0}-1\right)=\frac{\Delta\gamma}{\beta_0\gamma_0} \tag{4}$$

$$(1+\delta)^2=1+\frac{2p_t}{\beta_0}+p_t^2=\left(\frac{1}{\beta_0}+p_t\right)^2-\frac{1}{\beta_0^2\gamma_0^2} \tag{5}$$

$$\delta=\frac{p-p_0}{p_0}=\frac{\Delta p}{p_0}=\frac{p_t}{\beta}\approx\frac{p_t}{\beta_0} \tag{6}$$

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Using the Hamiltonian given in Eq. (1) we can derive 6×6 linear maps for various beam optical elements.
To get a linear transfer map, we first need to find the vector potential $\boldsymbol{A}$, expand the corresponding Hamiltonian to second order in dynamical variables and apply the Hamilton's equations. Then we can obtain linear second order differential equations, which can be solved and linear map can be obtained. In the following, we shall derive linear maps for various beam optical elements in synchrotrons.

Drift space

As a simplest case, let's first consider a drift (or free) space of length L. In this case $h = \frac{1}{\rho} = 0$ and $\boldsymbol{A} = 0$ so the Hamiltonian in Eq. (1) becomes

$$H = \frac{p_t}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + p_t\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2 - \frac{1}{\beta_0^2 \gamma_0^2}} \tag{7}$$

This Hamiltonian can be expanded to second order in dynamical variables and we get

$$H \approx -1 + \frac{p_x^2}{2p_0^2} + \frac{p_y^2}{2p_0^2} + \frac{p_t^2}{2\beta_0^2 \gamma_0^2} \tag{8}$$

The constant term can be neglected. Substituting this Hamiltonian into the Hamilton's equations, one can get the linear map, but here we show a different method.

The Hamilton's equations can be written in the form

$$\frac{dX}{ds} = S\nabla_{q,\pi} H \qquad (9)$$

where

$$X = \begin{pmatrix} x \\ \frac{\pi_x}{p_0} \\ y \\ \frac{\pi_y}{p_0} \\ z \\ p_t \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \qquad \nabla_{q,\pi} H = \begin{pmatrix} \partial H/\partial x \\ \partial H/\partial(\pi_x/p_0) \\ \partial H/\partial y \\ \partial H/\partial(\pi_y/p_0) \\ \partial H/\partial z \\ \partial H/\partial p_t \end{pmatrix} \qquad (10)$$

Note that $S^2 = -I$ and $S^T = -S$. $\qquad$ S^T: transpose of S

So the Hamilton's equations (9) can be expressed in matrix form:

$$\begin{pmatrix} dx/ds \\ d(\pi_x/p_0)/ds \\ dy/ds \\ d(\pi_y/p_0)/ds \\ dz/ds \\ dp_t/ds \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial x \\ \partial H/\partial(\pi_x/p_0) \\ \partial H/\partial y \\ \partial H/\partial(\pi_y/p_0) \\ \partial H/\partial z \\ \partial H/\partial p_t \end{pmatrix} \qquad (11)$$

If the Hamiltonian is a homogeneous polynomial of degree 2 [as in Eq. (8)], we can write

$$\nabla_{q,\pi} H = UX \qquad U\text{: a } 6 \times 6 \text{ matrix} \qquad (12)$$

Then the Hamilton's equations, Eq. (9) become

$$\frac{dX}{ds} = SUX \qquad (13) \qquad \text{so} \qquad \int_{X(0)}^{X(L)} \frac{dX}{X} = \int_0^L SU ds$$

The integrations lead to

$$X(L) = e^{LSU} X(0) \equiv M(L) X_0 \qquad \text{where} \quad X_0 = X(0) \qquad (14)$$

and

$$M(L) = e^{LSU} = \sum_{n=0}^{\infty} \frac{L^n}{n!} (SU)^n \qquad (15)$$

is the 6×6 linear map that we want.

Let's apply this observation to the drift space. With the expanded Hamiltonian given in Eq. (8), Eq. (12) becomes with Eq. (10)

$$\nabla_{q,\pi} H = \begin{pmatrix} 0 \\ p_x/p_0 \\ 0 \\ p_y/p_0 \\ 0 \\ p_t/\beta_0^2\gamma_0^2 \end{pmatrix} = UX = U \begin{pmatrix} x \\ p_x/p_0 \\ y \\ p_y/p_0 \\ z \\ p_t \end{pmatrix}$$

Note: For a drift space,

$$\pi_x = p_x$$
$$\pi_y = p_y$$

We find easily the matrix U by inspection

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/(\beta_0^2\gamma_0^2) \end{pmatrix}$$

Then Eq. (15) becomes

$$M(L) = e^{LSU} = \exp\left[L \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/(\beta_0^2\gamma_0^2) \end{pmatrix} \right]$$

Finally we get the linear map for drift space of length L:

$$M(L) \approx I + LSU = M_{\text{drift}} = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dfrac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (16)$$

where I is the unit matrix.

Note that Eq. (16) satisfies the symplectic condition $$M^T S M = S \qquad (17)$$

Note also that Eq. (17) is identical to $MSM^T = S$. By taking the determinant of Eq. (17), $\det(M^T S M) = \det(S)$, we find that $\det(M) = \pm 1$. Although not trivial to prove, it turns out that only $\det(M) = +1$ is valid. Read Appendix B for the derivation of Eq. (17).

Notice that if we apply the Hamilton's equation directly to the Hamiltonian before expansion in Eq. (6) we get the exact nonlinear map for the drift space:

$$\begin{aligned}
x_1 &= x_0 + \frac{p_{x0}/p_0}{d} L \\
p_{x1} &= p_{x0} \\
y_1 &= y_0 + \frac{p_{y0}/p_0}{d} L \\
p_{y1} &= p_{y0} \\
z_1 &= z_0 + \frac{L}{\beta_0}\left(1 - \frac{1}{d}\right) - \frac{L}{d} p_{t0} \\
p_{t1} &= p_{t0}
\end{aligned} \qquad (18)$$

where

$$d = \sqrt{\left(\frac{1}{\beta_0} + p_t\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2 - \frac{1}{\beta_0^2 \gamma_0^2}}$$

If we expand d^{-1} , substitute the result into Eq. (18) and keep the terms up to the first order in dynamical variables, we get the same linear map as Eq. (16).

Multipole expansion of two-dimensional fields

To obtain the Hamiltonian for an accelerator element, we have to know the vector potential. Assuming a magnet with infinite length, we can consider only two-dimensional transverse fields.

We are interested in the fields in source-free region (i.e. inside the vacuum chamber where particles move), then in that region magnetic fields must satisfy

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{and} \quad \nabla \times \boldsymbol{B} = 0 \tag{19}$$

With B_s constant (or zero), we can find B_x and B_y and write them in complex notation:

$$B(z) = B_y + iB_x = \sum_{n=0}^{\infty} C_n (x+iy)^n = \sum_{n=0}^{\infty} (B_n + iA_n)\,(x+iy)^n \tag{20}$$

where C_n is a complex constant, $C_n = B_n + iA_n$, with B_n, A_n being real. By applying the differential operator $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ to Eq. (20) we can show that Eq. (20) indeed satisfies Eq. (19).

Then we obtain the complex potential expanded in multipoles (in US convention):

$$A(z) = A_s + iV = -\sum_{n=0}^{\infty} \frac{1}{n+1} C_n (x+iy)^{n+1} \tag{21}$$

where V is the electric scalar potential. The complex potential and the complex fields are connected through

$$B(z) = -\frac{dA}{dz} \tag{22} \qquad (z = x+iy)$$

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From Eq. (21), we get the vector potential

$$A_s = -Re\sum_{n=0}^{\infty}\frac{1}{n+1}C_n(x+iy)^{n+1} = -Re\sum_{n=0}^{\infty}\frac{1}{n+1}(B_n+iA_n)(x+iy)^{n+1} \qquad (23)$$

We can rewrite Eq. (23) in the form

$$A_s = -\frac{p_0}{q}Re\sum_{n=0}^{\infty}\frac{1}{n+1}(b_n+ia_n)(x+iy)^{n+1} \qquad (24)$$

where b_n and a_n are normal and skew components of the multipole field respectively. They are related with the usual notation for multipole strength in a computer code like MAD:

$$b_n = \frac{q}{n!\,p_0}B_n = \frac{q}{n!\,p_0}\frac{\partial^n B_y}{\partial x^n}\bigg|_{(0,0,s)} = \frac{1}{n!}k_n \qquad k_n = \frac{q}{p_0}\frac{\partial^n B_y}{\partial x^n}\bigg|_{(0,0,s)}$$

$$a_n = \frac{q}{n!\,p_0}A_n = \frac{q}{n!\,p_0}\frac{\partial^n B_x}{\partial x^n}\bigg|_{(0,0,s)} = -\frac{1}{n!}\bar{k}_n \qquad \bar{k}_n = -\frac{q}{p_0}\frac{\partial^n B_x}{\partial x^n}\bigg|_{(0,0,s)} \qquad (25)$$

$n=0$: dipole

$n=1$: quadrupole

$n=2$: sextupole

$n=3$: octupole etc.

$$k_0 = \frac{qB_y}{p_0}\bigg|_{(0,0,s)} \qquad k_1 = \frac{q}{p_0}\frac{\partial B_y}{\partial x}\bigg|_{(0,0,s)}$$

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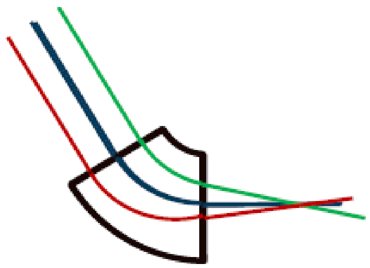
Sector bending (or dipole) magnet

For a dipole magnet, uniform magnetic field is given by $\boldsymbol{B}(x, y, s) = (0, B_0, 0)$. In curvilinear coordinate system, $\boldsymbol{B}$ is related to the vector potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \boldsymbol{e}_1 & h_2 \boldsymbol{e}_2 & h_3 \boldsymbol{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \qquad (26)$$



where $h_1 = h_2 = 1$, $h_3 = 1 + hx$, $(u_1, u_2, u_3) = (x, y, s)$, $(A_1, A_2, A_3) = \left(A_x, A_y, A_s\right)$

$\boldsymbol{e}_1 = \boldsymbol{e}_x$, $\boldsymbol{e}_2 = \boldsymbol{e}_y$, $\boldsymbol{e}_3 = \boldsymbol{e}_s$

Field components are then given by

$$B_x = \frac{\partial A_s}{\partial y} - \frac{1}{1 + hx}\frac{\partial A_y}{\partial s}, \qquad B_y = \frac{1}{1 + hx}\frac{\partial A_x}{\partial s} - \frac{\partial A_s}{\partial x} - \frac{hA_s}{1 + hx}, \qquad B_s = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad (27)$$

For dipole field, using a gauge, $A_x = A_y = 0$, the field $\boldsymbol{B} = B_0 \boldsymbol{e}_y$ can be derived from the vector potential

$$\boldsymbol{A} = (0, 0, -B_0 x + \frac{B_0 h x^2}{2(1 + hx)}) \qquad (28)$$

and the normalized vector potential is $\boldsymbol{a} = \frac{q}{p_0}\boldsymbol{A} = (0, 0, -k_0 x + \frac{k_0 h x^2}{2(1 + hx)})$ (29)

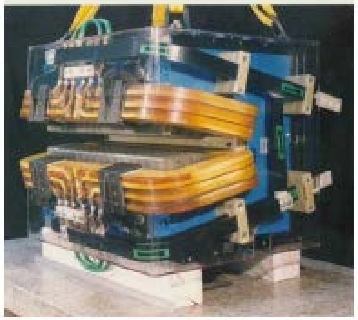
where $k_0 = \frac{q}{p_0} B_0$ (30)

Inserting Eq. (29) into the Hamiltonian given in Eq. (1), we have

$$H = \frac{p_t}{\beta_0} - (1+hx)\sqrt{\left(\frac{1}{\beta_0}+p_t\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - (1+hx)\left[-k_0 x + \frac{k_0 h x^2}{2(1+hx)}\right] \quad (31)$$

We can find an exact transformation from this Hamiltonian, but the result is not illuminating and it is sufficient to get the approximate linearized transformation. Hence, we expand Eq. (31) to second order in dynamical variables and get

$$H = \frac{1}{2}\left(\frac{p_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{p_y}{p_0}\right)^2 + \frac{p_t^2}{2\beta_0^2\gamma_0^2} + \left[k_0 - h\left(1+\frac{p_t}{\beta_0}\right)\right]x + \frac{k_0 h}{2}x^2 \quad (32$$



ALS 1.8 T dipole

For a combined-function magnet (i.e. bending magnet with normal quadrupole component in it) we have to include the focusing term. From Eqs. (24) and (25) we add the quadrupole focusing term to Eq. (32) to get

$$H = \frac{1}{2}\left(\frac{p_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{p_y}{p_0}\right)^2 + \frac{p_t^2}{2\beta_0^2\gamma_0^2} + \left[k_0 - h\left(1+\frac{p_t}{\beta_0}\right)\right]x + \frac{k_0 h}{2}x^2 + \frac{k_1}{2}(x^2 - y^2) \quad (33)$$

where $k_1 = \frac{q}{p_0}\frac{\partial B_y}{\partial x}\Big|_{(0,0,s)}$ is the normalized field gradient usually called the focusing strength.

With Eq. (33) we get the Hamilton's equations:

$$\frac{dx}{ds} = \frac{\partial H}{\partial(p_x/p_0)} = \frac{p_x}{p_0},$$

$$\frac{d(p_x/p_0)}{ds} = -\frac{\partial H}{\partial x} = -\left[k_0 - h\left(1 + \frac{p_t}{\beta_0}\right)\right] - k_0 h x - k_1 x$$

(34)

$$\frac{dy}{ds} = \frac{\partial H}{\partial(p_y/p_0)} = \frac{p_y}{p_0}, \qquad \frac{d(p_y/p_0)}{ds} = -\frac{\partial H}{\partial y} = k_1 y$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial p_t} = \frac{p_t}{\beta_0^2\gamma_0^2} - \frac{h}{\beta_0}x, \qquad \frac{dp_t}{ds} = -\frac{\partial H}{\partial z} = 0$$

The four first-order transverse equations can be combined to two second-order differential equations:

$$x'' + K_x x = \frac{hp_t}{\beta_0} + (h - k_0)$$

$$y'' - k_1 y = 0 \qquad (35)$$

where $K_x = hk_0 + k_1$.

$$x' = \frac{dx}{ds} = \frac{p_x}{p_0} \qquad y' = \frac{dy}{ds} = \frac{p_y}{p_0}$$

$$k_0 = \frac{q}{p_0}B_0 \qquad k_1 = \frac{q}{p_0}\frac{\partial B_y}{\partial x}\bigg|_{(0,0,s)}$$

With the general initial conditions, i.e. at $s = 0, x = x_0,\ \ y = y_0, x' = x_0',\ y' = y_0',$ and $z = z_0, p_t = p_{t0}$, the solutions can be obtained easily. For example, the horizontal equation can be solved using the variation of parameters (or Green's function) method.

The solutions are given by

$$x(L) = x_0 \cos\sqrt{K_x}L + x_0' \frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} + p_{t0}\frac{h}{\beta_0}\frac{1-\cos\sqrt{K_x}L}{K_x} + (h-k_0)\frac{1-\cos\sqrt{K_x}L}{K_x}$$

$$x'(L) = -x_0\sqrt{K_x}\sin\sqrt{K_x}L + x_0'\cos\sqrt{K_x}L + p_{t0}\frac{h}{\beta_0}\frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} + (h-k_0)\frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} \tag{36}$$

$$y(L) = y_0\cosh\sqrt{k_1}L + y_0'\frac{\sinh\sqrt{k_1}L}{\sqrt{k_1}}$$

$$y'(L) = y_0\sqrt{k_1}\sinh\sqrt{k_1}L + y_0'\cosh\sqrt{k_1}L$$

$$x' = p_x/p_0$$

$$y' = p_y/p_0$$

where L is the effective (or magnetic) length of the dipole magnet, which is the path length of the reference particle inside the magnet.

With $x(s)$ in the above the longitudinal equation in Eq. (34) can be integrated to yield

$$z(L) = -\frac{h}{\beta_0}\frac{\sin\sqrt{K_x}L}{\sqrt{K_x}}x_0 - \frac{h}{\beta_0}\frac{1-\cos\sqrt{K_x}L}{K_x}x_0' + z_0 + \frac{L}{\beta_0^2\gamma_0^2}p_{t0} - \frac{h^2}{\beta_0^2 K_x}\left(L - \frac{1}{\sqrt{K_x}}\sin\sqrt{K_x}L\right)p_{t0}$$
$$-\frac{h}{\beta_0}\frac{h-k_0}{K_x}\left(L - \frac{1}{\sqrt{K_x}}\sin\sqrt{K_x}L\right) \tag{37}$$

$$p_t = p_{t0}$$

These solutions can be written in the form of a matrix equation:

$$X = M_{bend}X_0 + m \tag{38}$$

where

$$M_{bend}=\begin{pmatrix} \cos\sqrt{K_x}L & \frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} & 0 & 0 & 0 & \frac{h}{\beta_0}\frac{1-\cos\sqrt{K_x}L}{K_x} \\ -\sqrt{K_x}\sin\sqrt{K_x}L & \cos\sqrt{K_x}L & 0 & 0 & 0 & \frac{h}{\beta_0}\frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} \\ 0 & 0 & \cosh\sqrt{k_1}L & \frac{\sinh\sqrt{k_1}L}{\sqrt{k_1}} & 0 & 0 \\ 0 & 0 & \sqrt{k_1}\sinh\sqrt{k_1}L & \cosh\sqrt{k_1}L & 0 & 0 \\ -\frac{h}{\beta_0}\frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} & -\frac{h}{\beta_0}\frac{1-\cos\sqrt{K_x}L}{K_x} & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2}-\frac{h^2}{\beta_0^2}\frac{L-\frac{1}{\sqrt{K_x}}\sin\sqrt{K_x}L}{K_x} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (39)$$

and the vector m which is the zeroth-order solution and X_0 are given by

$$m=\begin{pmatrix} (h-k_0)\frac{1-\cos\sqrt{K_x}L}{K_x} \\ (h-k_0)\frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} \\ 0 \\ 0 \\ -\frac{h}{\beta_0}\frac{h-k_0}{K_x}\left(L-\frac{1}{\sqrt{K_x}}\sin\sqrt{K_x}L\right) \\ 0 \end{pmatrix} \quad (40) \qquad X_0=\begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ p_{t0} \end{pmatrix} \quad (41)$$

Usually, the reference orbit curvature h is adjusted to the reference momentum p_0, by demanding horizontal kick to vanish [in Eq. (36)] under the condition $x = 0, p_x = 0$, and $p_t = 0$, i.e.

$$h = \frac{1}{\rho} = \frac{qB_0}{p_0} = k_0 \tag{42}$$

Then the Hamiltonian for linear motion in bending magnet, matched to the reference momentum p_0 is

$$H = \frac{1}{2}\left(\frac{p_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{p_y}{p_0}\right)^2 + \frac{p_t^2}{2\beta_0^2\gamma_0^2} - \frac{hp_t}{\beta_0}x + \frac{h^2}{2}x^2 + \frac{k_1}{2}(x^2 - y^2) \tag{43}$$

In this case the zeroth-order matrix m is zero and

$$K_x = hk_0 + k_1 = h^2 + k_1 = \frac{1}{\rho^2} + \frac{B'}{B_0\rho} = \frac{1-n}{\rho^2} \tag{44}$$

where we have introduced the field index

$$n = -\frac{\rho}{B_0}\frac{\partial B_y}{\partial x} = -\rho^2 k_1 \tag{45}$$

which is a dimensionless quantity. A dipole magnet with positive(negative) field index leads to the vertical focusing(defocusing).

In Eq. (44), $B\rho = \frac{p}{q}$ is a quantity called the (magnetic) rigidity, which is a measure of the resistance of a particle to deflection by magnetic fields. In practical units, with the particle energy E in GeV unit, the rigidity is given by

$$B\rho[T \cdot m] \approx 3.3356\, E[GeV] \tag{46}$$

Dipole fringe fields and edge focusing

There can be a significant (and complicated) effect when a particle enters or leaves the region near the magnet boundary (edge). This region is called the fringe (field) region and should be taken into account properly. Let the vertical field at the main body be $B_y = B_0$.

Assume the vertical field B_y to vary linearly in the fringe region $0 < s < d_f$. Then the field in this region is given by

$$B_x = 0, \quad B_y = \frac{s}{d_f} B_0, \quad B_s = \frac{y}{d_f} B_0 \tag{47}$$

where we neglect B_x assuming an infinite magnet width and B_s is required to satisfy Maxwell's equation, $(1 + hx)\frac{\partial B_s}{\partial y} \approx \frac{\partial B_s}{\partial y} = \frac{\partial B_y}{\partial s}$ assuming $hx \ll 1$.

Let's now assume the entrance (or exit) face of a dipole is at $s = s_0$, and the pole face is rotated clockwise by ψ_1 in the bending plane about the y axis (see Figure in the next page). This has three effects to take into account: 1) The fringe field extends over a distance L of the reference orbit $(L > d_f)$

$$L = \frac{d_f}{\cos \psi_1} \tag{48}$$

2) The field itself is rotated with the face of the magnet, so in the fringe fringe field region $0 < s < L$,

$$B_x = -\frac{y}{d_f} B_0 \sin \psi_1 = -\frac{y}{L} B_0 \tan \psi_1, \quad B_s = \frac{y}{d_f} B_0 \cos \psi_1 = \frac{y}{L} B_0 \tag{49}$$

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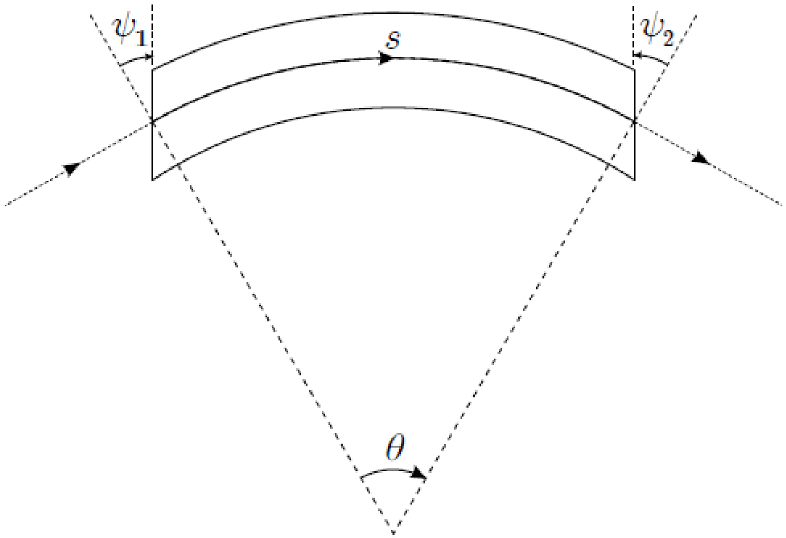

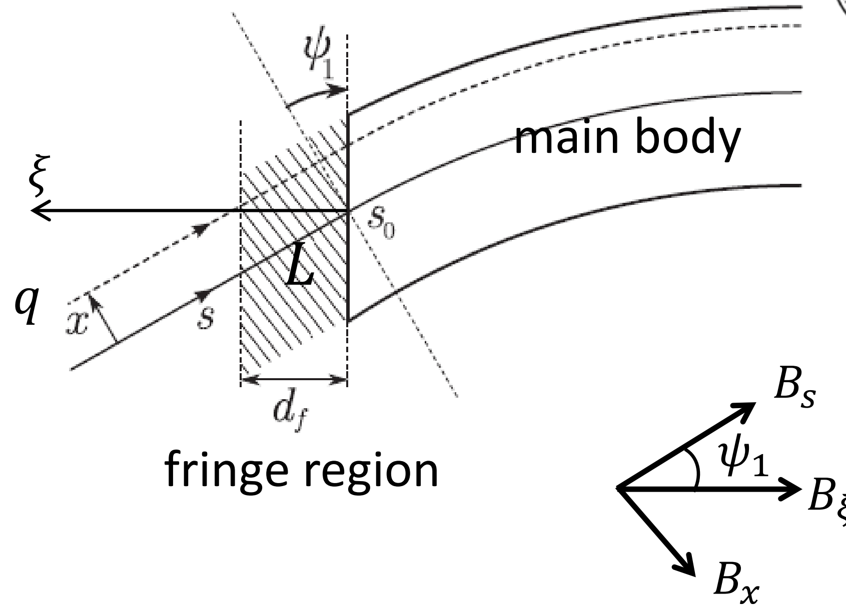

Note that if $\psi_1 \to 0$, Eq. (49) becomes Eq. (47) as it should be.

3) Particles with different x coordinates enter the main field of the magnet at different s positions. A particle with coordinate x travelling parallel to the reference orbit passes the entrance face at $s = s_0 + x \tan \psi_1$. This particle therefore sees a defit integrated vertical field $B_0 x \tan \psi_1$, so that in the region $0 < s < L$ leading up to the entrance of the dipole, the vertical field component is given by

$$B_y = \frac{s}{L} B_0 - \frac{x}{L} B_0 \tan \psi_1 \qquad (50)$$

Eqs. (49) and (50) satisfy the Maxwell's equations, $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = 0$.

With the field component given by Eqs. (49) and (50) and the gauge $A_x = 0$ we can obtain the vector potential from $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

$$\boldsymbol{A} = \left(0, \quad \frac{B_0}{L} xy, \quad \frac{B_0}{2L}(x^2 - y^2)\tan\psi_1 - \frac{B_0}{L} xs\right) \tag{51}$$

Let's revisit the Hamiltonian given by Eq. (1):

$$H = \frac{p_t}{\beta_0} - (1 + hx)\sqrt{\left(\frac{1}{\beta_0} + p_t\right)^2 - \left(\frac{\pi_x}{p_0} - a_x\right)^2 - \left(\frac{\pi_y}{p_0} - a_y\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - (1 + hx)a_s$$

Inserting the normalized vector potential

$$\boldsymbol{a} = \frac{q}{p_0}\boldsymbol{A} = \left(0, \quad \frac{q}{p_0}\frac{B_0}{L} xy, \quad \frac{q}{p_0}\frac{B_0}{2L}(x^2 - y^2)\tan\psi_1 - \frac{q}{p_0}\frac{B_0}{L} xs\right) \tag{52}$$

and expanding the result to second order in canonical variables, neglecting the constant term we have

$$H \approx \frac{1}{2}\left(\frac{\pi_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{\pi_y}{p_0} - \frac{q}{p_0}\frac{B_0}{L} xy\right)^2 - (1 + hx)\left[\frac{qB_0}{2p_0L}(x^2 - y^2)\tan\psi_1 - \frac{q}{p_0}\frac{B_0}{L} xs\right] + \frac{p_t^2}{2\beta_0^2\gamma_0^2}$$

$$\approx \frac{1}{2}\left(\frac{\pi_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{\pi_y}{p_0}\right)^2 - \frac{qB_0}{2p_0L}(x^2 - y^2)\tan\psi_1 + \frac{q}{p_0}\frac{B_0}{L} xs + \frac{p_t^2}{2\beta_0^2\gamma_0^2} \tag{53}$$

Introducing the parameter

$$k_f = \frac{q}{p_0}\frac{B_0}{L}\tan\psi_1 \tag{54}$$

the Hamiltonian can be written as

$$H \approx \frac{1}{2}\left(\frac{\pi_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{\pi_y}{p_0}\right)^2 - \frac{k_f}{2}(x^2 - y^2) + k_f \cot\psi_1\, xs + \frac{p_t^2}{2\beta_0^2\gamma_0^2} \tag{55}$$

With this Hamiltonian we obtain the Hamilton's equations

$$\frac{dx}{ds} = \frac{\partial H}{\partial(\pi_x/p_0)} = \frac{\pi_x}{p_0}, \qquad \frac{d(\pi_x/p_0)}{ds} = -\frac{\partial H}{\partial x} = k_f x - k_f \cot\psi_1\, s$$

$$\frac{dy}{ds} = \frac{\partial H}{\partial(\pi_y/p_0)} = \frac{\pi_y}{p_0}, \qquad \frac{d(\pi_y/p_0)}{ds} = -\frac{\partial H}{\partial y} = -k_f y \tag{56}$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial p_t} = \frac{p_t}{\beta_0^2\gamma_0^2}, \qquad \frac{dp_t}{ds} = -\frac{\partial H}{\partial z} = 0$$

The last two longitudinal equations can be easily integrated to yield

$$z(L) = z(0) + \frac{L}{\beta_0^2\gamma_0^2}p_t(0)$$

$$p_t(L) = p_t(0) \tag{57}$$

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The four transverse equations can be combined to two second-order differential equations:

$$x'' - k_f x = -k_f \cot\psi_1 \, s$$
$$y'' + k_f y = 0 \tag{58}$$

The solutions to the vertical equation are

$$y(L) = y_0 \cos\sqrt{k_f}L + y_0' \frac{\sin\sqrt{k_f}L}{\sqrt{k_f}}$$
$$y'(L) = -y_0\sqrt{k_f}\sin\sqrt{k_f}L + y_0'\cos\sqrt{k_f}L \tag{59}$$

The horizontal equation can be solved by variation of parameters method. The solutions are

$$x(L) = x_0 \cosh\sqrt{k_f}L + x_0' \frac{\sinh\sqrt{k_f}L}{\sqrt{k_f}} + \left(L - \frac{\sinh\sqrt{k_f}L}{\sqrt{k_f}}\right)\cot\varepsilon \qquad \varepsilon = \psi_1$$
$$x'(L) = x_0\sqrt{k_f}\sinh\sqrt{k_f}L + x_0'\cosh\sqrt{k_f}L + \left(1 - \cosh\sqrt{k_f}L\right)\cot\varepsilon \tag{60}$$

where we have changed the notation of the edge angle from ψ_1 to ε following the convention adopted by many textbooks.

Eqs. (57), (59) and (60) are the linear transfer map for a fringe region of length L for a particle entering (or leaving) the magnet pole face at an angle ε.

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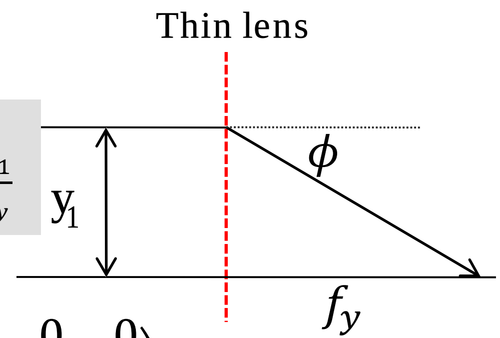
The fringe field extent L depends on the magnet structure. To avoid such ambiguity, it is customary to take thin-lens approximation, i.e. $L \to 0$.

For $L \to 0$, these become

$$
\begin{aligned}
x(L) &= x_0 \\
x'(L) &= x_0 k_f L + x_0' = x_0 k_f \tan\varepsilon + x_0' = x_0 \frac{\tan\varepsilon}{\rho} + x_0' \qquad [\because \text{Eq. (54)}] \\
y(L) &= y_0 \\
y'(L) &= -y_0 k_f L + y_0' = -y_0 k_f \tan\varepsilon + y_0' = -y_0 \frac{\tan\varepsilon}{\rho} + y_0' \\
z(L) &= z_0 \\
p_t(L) &= p_{t0}
\end{aligned}
\tag{61}
$$

$$
y = y_1
$$

$$
y' = y_1' - \tan\phi = y_1' - \frac{y_1}{f_y}
$$



Thus, we have the matrix element for edge focusing

$$
M_{edge} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tan\varepsilon}{\rho} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\tan\varepsilon}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{f_x} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{f_y} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{62}
$$

Note that there is a vertical focusing and horizontal defocusing when the edge angle ε is positive and vertical defocusing and horizontal focusing when the edge angle is negative.

For an arbitrary wedge magnet with entrance angle ε_1 and exit angle ε_2, the linear map is given by

$$M_{wedge} = M_{\varepsilon_2} M_{bend} M_{\varepsilon_1}$$

$$M_{wedge} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tan\varepsilon_2}{\rho} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\tan\varepsilon_2}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\sqrt{K_x}L & \frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} & 0 & 0 & 0 & \frac{h}{\beta_0}\frac{1-\cos\sqrt{K_x}L}{K_x} \\ -\sqrt{K_x}\sin\sqrt{K_x}L & \cos\sqrt{K_x}L & 0 & 0 & 0 & \frac{h}{\beta_0}\frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} \\ 0 & 0 & \cosh\sqrt{k_1}L & \frac{\sinh\sqrt{k_1}L}{\sqrt{k_1}} & 0 & 0 \\ 0 & 0 & \sqrt{k_1}\sinh\sqrt{k_1}L & \cosh\sqrt{k_1}L & 0 & 0 \\ -\frac{h}{\beta_0}\frac{\sin\sqrt{K_x}L}{\sqrt{K_x}} & -\frac{h}{\beta_0}\frac{1-\cos\sqrt{K_x}L}{K_x} & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} - \frac{h^2}{\beta_0^2}\frac{L-\frac{1}{\sqrt{K_x}}\sin\sqrt{K_x}L}{K_x} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tan\varepsilon_1}{\rho} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\tan\varepsilon_1}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (63)$$

where $K_x = hk_0 + k_1 = h^2 + k_1 = \frac{1}{\rho^2} + \frac{B'}{B_0\rho} = \frac{1-n}{\rho^2}$

The combined matrix after the matrix multiplications is complicated in form and can be obtained with the help of MATHEMATICA.

In the case of a uniform rectangular magnet of bending angle $\theta(= L/\rho)$, the entrance and exit angles are the same given by, $\varepsilon_1 = \varepsilon_2 = \varepsilon = \theta/2$. The linear transfer map is

$$M_{rect} = \begin{pmatrix} 1 & \rho\sin\theta & 0 & 0 & 0 & \dfrac{\rho(1-\cos\theta)}{\beta} \\ 0 & 1 & 0 & 0 & 0 & \dfrac{2\tan\frac{\theta}{2}}{\beta} \\ 0 & 0 & \cos\theta & \rho\sin\theta & 0 & 0 \\ 0 & 0 & -\dfrac{\sin\theta}{\rho} & \cos\theta & 0 & 0 \\ -\dfrac{2\tan\frac{\theta}{2}}{\beta} & -\dfrac{\rho(1-\cos\theta)}{\beta} & 0 & 0 & 1 & -L+\dfrac{\rho\sin\theta}{\beta^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (64)$$

In the case of a uniform rectangular magnet, there is no focusing in the horizontal plane while there is a weak focusing in the vertical plane.

3. Linear Maps II

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Quadrupole magnet

The vector potential A_s of a normal quadrupole magnet can be obtained from $n = 1$ in Eqs. (24) and (25)

$$\frac{q}{p_0}\boldsymbol{A} = \boldsymbol{a} = \left(0,0,-\frac{1}{2}k_1(x^2 - y^2)\right) \quad (65)$$

The corresponding scaled magnetic field is given by

$$\frac{q}{p_0}\boldsymbol{B} = (k_1 y,\ k_1 x, 0) \quad (66)$$

$$k_1 = b_1 = \frac{q}{p_0}\frac{\partial B_y}{\partial x} = \frac{q}{p_0}\frac{\partial B_x}{\partial y}$$ quadrupole (focusing) strength

Inserting Eq. (65) into the Hamiltonian, Eq. (1), and letting $h \to 0$, we get the Hamiltonian for a normal quadrupole

$$H = \frac{p_t}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + p_t\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{1}{2}k_1(x^2 - y^2) \quad (67)$$

Expanding to second order and neglecting the constant term, we find

$$H \approx \frac{1}{2}\left(\frac{p_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{p_y}{p_0}\right)^2 + \frac{k_1}{2}(x^2 - y^2) + \frac{p_t^2}{2\beta_0^2\gamma_0^2} \quad (68)$$

The Hamilton's equations are

$$\frac{dx}{ds} = \frac{\partial H}{\partial (p_x/p_0)} = \frac{p_x}{p_0}, \qquad \frac{d(p_x/p_0)}{ds} = -\frac{\partial H}{\partial x} = -k_1 x$$

$$\frac{dy}{ds} = \frac{\partial H}{\partial (p_y/p_0)} = \frac{p_y}{p_0}, \qquad \frac{d(p_y/p_0)}{ds} = -\frac{\partial H}{\partial y} = k_1 y \qquad (69)$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial p_t} = \frac{p_t}{\beta_0^2 \gamma_0^2}, \qquad \frac{dp_t}{ds} = -\frac{\partial H}{\partial z} = 0$$

By similar method introduced before, we can find easily the linear map for a quadrupole of effective length L

$$M_Q = \begin{pmatrix} \cos\sqrt{k_1}L & \frac{1}{\sqrt{k_1}}\sin\sqrt{k_1}L & 0 & 0 & 0 & 0 \\ -\sqrt{k_1}\sin\sqrt{k_1}L & \cos\sqrt{k_1}L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\sqrt{k_1}L & \frac{1}{\sqrt{k_1}}\sinh\sqrt{k_1}L & 0 & 0 \\ 0 & 0 & \sqrt{k_1}\sinh\sqrt{k_1}L & \cosh\sqrt{k_1}L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (70)$$

If we take $L \to 0$ the map for the horizontal focusing quadrupole magnet in thin-lens approximation can be obtained

$$M_{thin\ Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{71}$$

where

$$f = \frac{1}{k_1 L} = \frac{p_0}{qLB'} \tag{72}$$

is the focal length of a quadrupole magnet.

Note that if a quadrupole magnet focuses a beam in the horizontal (vertical) plane then it defocuses in the vertical (horizontal) plane.

To achieve focusing in both planes, quadrupole doublet or triplet is employed.

In the following we briefly analyze the quadrupole doublet. The case for triplet can be similarly applied.

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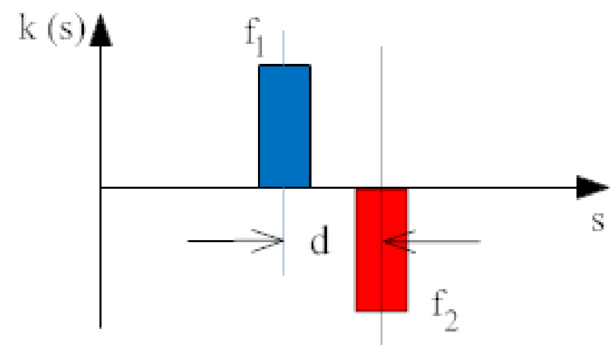
Let's consider quadrupole doublet in the thin-lens approx.

The transfer matrix in one plane is given by

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{d}{f_1} & d \\ -\frac{1}{f^*} & 1-\frac{d}{f_2} \end{pmatrix}$$

where

$$\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \qquad (73)$$



Let $f_1 = -f_2 = f$ $\Longrightarrow$ $M_{FD} = \begin{pmatrix} 1-\frac{d}{f} & d \\ -\frac{d}{f^2} & 1+\frac{d}{f} \end{pmatrix}$ $\Longrightarrow$ $M_{FD} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{d}{f^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}$

(F-D)

$$\frac{1}{f^*} = \frac{d}{f^2} > 0$$

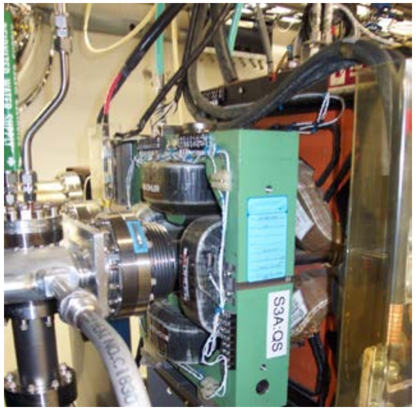
focusing

The matrix in the other plane (D-F) is

$$M_{DF} = \begin{pmatrix} 1+\frac{d}{f} & d \\ -\frac{d}{f^2} & 1-\frac{d}{f} \end{pmatrix} = \begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{d}{f^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$$

focusing

Skew quadrupole magnet



A skew quadrupole is obtained from a normal quadrupole by rotating the magnet by 45° about the magnetic axis.

There are two methods to get the linear map for a skew quadupole.

The first method follows just our standard procedure as before; get the vector potential for a skew quad and substitute it into the expanded Hamiltonian. The skew multipole vector potential components are given by the a_n coefficients in the multipole expansion Eq. (24)

$$A_s = -\frac{p_0}{q} Re \sum_{n=0}^{\infty} \frac{1}{n+1} (b_n + i a_n)(x+iy)^{n+1}$$

For a skew quadrupole, all coefficients are zero except for a_1: $\frac{q}{p_0} A_s = a_1 xy \equiv -\bar{k}_1 xy$

$$\bar{k}_1 = -a_1 = -\frac{q}{p_0}\frac{\partial B_x}{\partial x} = \frac{q}{p_0}\frac{\partial B_y}{\partial y}$$

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The Hamiltonian for a skew quadrupole magnet is then

$$H = \frac{p_t}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + p_t\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \bar{k}_1 xy \qquad (74)$$

The Hamiltonian expanded to second order becomes

$$H \approx \frac{1}{2}\left(\frac{p_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{p_y}{p_0}\right)^2 + \bar{k}_1 xy + \frac{p_t^2}{2\beta_0^2\gamma_0^2} \qquad (75)$$

Note the term with xy: this term leads to coupling of the horizontal and vertical motion. The skew quadrupole gives a horizontal kick proportional to the vertical offset of the particle, and vice-versa.

Hamilton's equations with the second-order skew quadrupole Hamiltonian Eq. (75) may be solved as for the normal quadrupole. The procedure can be found in Wiedemann's book (H. Wiedemann, Particle accelerator physics, Springer 4th ed). But here we shall take a simpler way. A skew quadrupole can be obtained from a normal quadrupole by rotating 45 degree with respect to the longitudinal axis:

$$M_{SQ} = R\left(\frac{\pi}{4}\right) M_Q R\left(-\frac{\pi}{4}\right) \qquad (76)$$

where M_Q is the transfer matrix for a normal quadrupole given in Eq. (70) and $R(\theta)$ is the rotation matrix by an angle θ w. r. t. the magnet axis. $R(\theta)$ is given by

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$$R(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{77}$$

Matrix multiplications in Eq. (76) lead to

$$M_{SQ} = \frac{1}{2}\begin{pmatrix} C^+ & \frac{1}{\sqrt{\bar{k}_1}}S^+ & C^- & \frac{1}{\sqrt{\bar{k}_1}}S^- & 0 & 0 \\ -\sqrt{\bar{k}_1}S^- & C^+ & -\sqrt{\bar{k}_1}S^+ & C^- & 0 & 0 \\ C^- & \frac{1}{\sqrt{\bar{k}_1}}S^- & C^+ & \frac{1}{\sqrt{\bar{k}_1}}S^+ & 0 & 0 \\ -\sqrt{\bar{k}_1}S^- & C^- & -\sqrt{\bar{k}_1}S^- & C^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{78}$$

where

$$C^{\pm} = \cos\sqrt{\bar{k}_1}L \pm \cosh\sqrt{\bar{k}_1}L$$

$$S^{\pm} = \sin\sqrt{\bar{k}_1}L \pm \sinh\sqrt{\bar{k}_1}L$$

$$\bar{k}_1 = -\frac{q}{p_0}\frac{\partial B_x}{\partial x} = \frac{q}{p_0}\frac{\partial B_y}{\partial y}$$

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Skew quadrupoles are used to control the beam emittance (sizes) in synchrotrons and storage rings. In third-generation synchrotron radiation sources, skew quadrupoles control (minimize) the vertical emittance such that the beam brightness is maximized. In future light sources like 4GSR, skew quads can be used to make a round beam.

In PLS-2 storage ring, there are 24 skew quads distributed around the ring, which can control the horizontal and vertical coupling of emittance and as a result the vertical beam emittance is a few percent of the horizontal beam emittance.

Note: beam emittance is defined as the phase space area occupied by a beam divided by π.
We shall discuss about the beam emittance in the next Lecture.

RF cavity

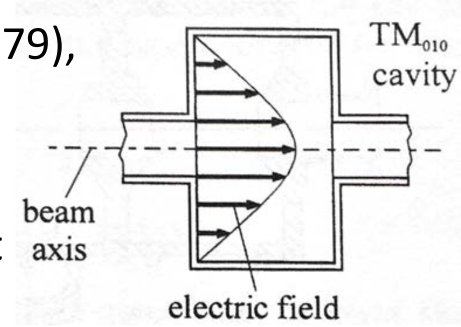
Let's assume that an RF cavity is of cylindrical type of TM_{010} mode. Then the fields are of the form (e.g. Jackson, Chap. 8)

$$E_s = -(\nabla\Phi)_s - \frac{\partial A_s}{\partial t} = -\frac{\partial A_s}{\partial t} = E_0 J_0(kr)\sin(\omega_{RF} t + \phi_0) \qquad (79)$$

$$B_\theta = (\nabla \times \boldsymbol{A})_\theta = -\frac{\partial A_r}{\partial s} - \frac{\partial A_s}{\partial r} = -\frac{\partial A_s}{\partial r} = \frac{E_0}{c} J_1(kr)\cos(\omega_{RF} t + \phi_0) \qquad (80)$$

where $r = \sqrt{x^2 + y^2}$. All other field components are zero. From Eq. (79), we find

$$A_s = -\int E_s dt = \frac{E_0 J_0(kr)}{\omega_{RF}} \cos(\omega_{RF} t + \phi_0) \qquad (81)$$



with $A_x = A_y = 0$. Here $\omega_{RF} = ck$ and $k = \frac{p_{01}}{a}$ with p_{01} the first root of $J_0(ka) = 0$ (bounbary condition) where a is the cavity radius ($p_{01} \approx 2.405$). Then the Hamiltonian Eq. (1) for a RF cavity is given by

$$H = \frac{p_t}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + p_t\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - \frac{q}{p_0}\frac{E_0}{\omega_{RF}} J_0(kr)\cos\left(\frac{k}{\beta_0}s - kz + \phi_0\right) \qquad (82)$$

where we have changed the variable from time t to $z = \frac{s}{\beta_0} - ct$.

Note that this Hamiltonian for a RF cavity depends on the longitudinal variable s, which usually is not easy to integrate the equations of motion. For simplicity, we assume that it is possible to average the Hamiltonian over the length of the cavity:

$$\langle H\rangle = \frac{1}{L}\int_{-L/2}^{L/2} H ds \tag{83}$$

Let

$$H = H_0 + H_1$$

where H_1 is the term that depends on the longitudinal variable s, i.e.

$$H_1 = -\frac{q}{p_0}\frac{E_0}{\omega_{RF}} J_0(kr)\cos\left(\frac{k}{\beta_0}s - kz + \phi_0\right)$$

Averaging this we get

$$\langle H_1\rangle = -\frac{1}{L}\frac{q}{p_0}\frac{E_0}{\omega_{RF}} J_0(kr)\int_{-L/2}^{L/2}\cos\left(\frac{k}{\beta_0}s - kz + \phi_0\right) ds = -\frac{1}{L}\frac{q}{p_0}\frac{E_0}{\omega_{RF}} J_0(kr)\frac{2\beta_0}{k}\sin\frac{kL}{2\beta_0}\cos(-kz+\phi_0)$$

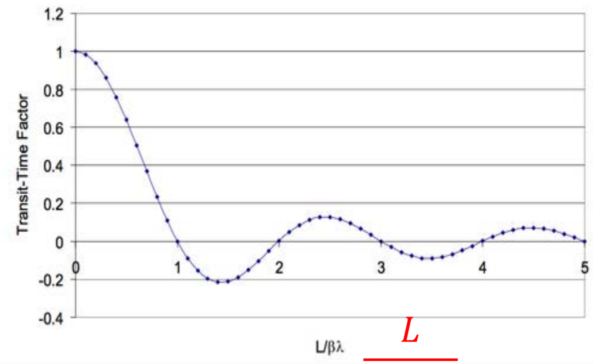
$$\equiv -\frac{\alpha}{\pi} J_0(kr)\cos(-kz+\phi_0)$$

where

$$\alpha = \frac{\pi}{L}\frac{q}{p_0}\frac{E_0}{\omega_{RF}} J_0(kr)\frac{2\beta_0 E_0}{ck^2}\sin\frac{kL}{2\beta_0} \equiv \frac{\pi q}{p_0}\frac{E_0}{\omega_{RF}} T \tag{84}$$

Here T is the transit-time factor (dimensionless) given by

$$T = \frac{\sin\frac{kL}{2\beta_0}}{\frac{kL}{2\beta_0}} \tag{85}$$



$\frac{L}{\beta_0\lambda_{RF}}$

which is a parameter introduced to take into account the variation in the electric field over the time taken for a particle to pass through the cavity.

Although the fields for TM_{010} mode do not depend on the length of the cavity, it is customary to choose the cavity length to $L = \frac{\lambda_{RF}}{2} = \frac{\pi}{k}$.

The cavity voltage V_0 is defined in terms of the electric field amplitude E_0, and the length of the cavity L such that

$$\frac{V_0}{L} = E_0 T \quad (86)$$

So

$$\alpha = \frac{qV_0}{p_0 c} \quad (87)$$

The averaged Hamiltonian then becomes

$$\langle H \rangle = \frac{p_t}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + p_t\right)^2 - \left(\frac{p_x}{p_0}\right)^2 - \left(\frac{p_y}{p_0}\right)^2 - \frac{1}{\beta_0^2 \gamma_0^2}} - \frac{\alpha}{\pi} J_0(kr)\cos(-kz + \phi_0) \quad (88)$$

$r = \sqrt{x^2 + y^2}$

We now expand the Hamiltonian to second order in dynamical variables. Using

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{6} - \cdots \quad (89)$$

and dropping the constant term, we get the averaged and expanded Hamiltonian

$$H = \frac{1}{2}\left(\frac{\pi_x}{p_0}\right)^2 + \frac{1}{2}\left(\frac{\pi_y}{p_0}\right)^2 + \frac{p_t^2}{2\beta_0^2 \gamma_0^2} + \frac{\alpha}{4\pi} k^2 \,(x^2 + y^2)\cos\phi_0 - \frac{\alpha}{\pi} kz\, \sin\phi_0 + \frac{\alpha}{2\pi} k^2 z^2 \cos\phi_0 \quad (90)$$

The Hamilton's equations are

$$\frac{dx}{ds} = \frac{\partial H}{\partial(\pi_x/p_0)} = \frac{\pi_x}{p_0} = \frac{p_x}{p_0}, \qquad \frac{d(\pi_x/p_0)}{ds} = -\frac{\partial H}{\partial x} = -\frac{\alpha}{2\pi}k^2\cos\phi_0 x$$

$$\frac{dy}{ds} = \frac{\partial H}{\partial(\pi_y/p_0)} = \frac{\pi_y}{p_0} = \frac{p_y}{p_0}, \qquad \frac{d(\pi_y/p_0)}{ds} = -\frac{\partial H}{\partial y} = -\frac{\alpha}{2\pi}k^2\cos\phi_0 y \qquad (91)$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial p_t} = \frac{p_t}{\beta_0^2\gamma_0^2}, \qquad \frac{dp_t}{ds} = -\frac{\partial H}{\partial z} = -\frac{\alpha}{\pi}k^2\cos\phi_0 z$$

The four transverse equations can be combined to yield

$$x'' = -\frac{\alpha}{2\pi}k^2\cos\phi_0 x \equiv -K_\perp^2 x$$

$$y'' = -\frac{\alpha}{2\pi}k^2\cos\phi_0 y \equiv -K_\perp^2 y \qquad (92)$$

where $K_\perp^2 = \frac{\alpha}{2\pi}k^2\cos\phi_0$ (93)

The two longitudinal equations in Eq. (91) lead to

$$z'' + K_\parallel^2 z = -\frac{1}{\beta_0^2\gamma_0^2}\frac{\alpha}{\pi}k^2\cos\phi_0 \qquad (94)$$

where $K_\parallel^2 = \frac{1}{\beta_0^2\gamma_0^2}\frac{\alpha}{\pi}k^2\cos\phi_0$ (95)

With general initial conditions, these equations can be solved easily and the result can be expressed in matrix form

$$X = M_{cavity} X_0 + m \qquad (96)$$

where

$$M_{cavity} = \begin{pmatrix} \cos K_\perp L & \frac{1}{K_\perp}\sin K_\perp L & 0 & 0 & 0 & 0 \\ -K_\perp \sin K_\perp L & \cos K_\perp L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos K_\perp L & \frac{1}{K_\perp}\sin K_\perp L & 0 & 0 \\ 0 & 0 & -K_\perp \sin K_\perp L & \cos K_\perp L & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos K_\parallel L & \frac{1}{\beta_0^2\gamma_0^2 K_\parallel}\sin K_\parallel L \\ 0 & 0 & 0 & 0 & -\beta_0^2\gamma_0^2 K_\parallel \sin K_\parallel L & \cos K_\parallel L \end{pmatrix} \qquad (97)$$

and the zeroth-order solution is given by the vector m:

$$m = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{2L\tan\phi_0}{\pi}\sin^2\frac{K_\parallel L}{2} \\ \alpha \sin\phi_0 \frac{\sin K_\parallel L}{K_\parallel L} \end{pmatrix} \qquad (98)$$

For small α (i.e. high energy particle in a cavity with a weak field), $K_{\parallel}L \ll 1$ so in this case M_{cavity} becomes

$$M_{cavity} = \begin{pmatrix} \cos K_{\perp}L & \frac{1}{K_{\perp}}\sin K_{\perp}L & 0 & 0 & 0 & 0 \\ -K_{\perp}\sin K_{\perp}L & \cos K_{\perp}L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos K_{\perp}L & \frac{1}{K_{\perp}}\sin K_{\perp}L & 0 & 0 \\ 0 & 0 & -K_{\perp}\sin K_{\perp}L & \cos K_{\perp}L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & -\frac{qV_0}{p_0c}\frac{\pi}{L}\cos\phi_0 & 1 \end{pmatrix} \quad (99)$$

and all the zeroth-order transfer map m for an RF cavity can be neglected.
In the thin-lens ($L \to 0$) and weak field approximation, this becomes

$$M_{cavity} \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{f_{\parallel}} & 1 \end{pmatrix} \quad (100)$$

Identity transform transversely and focusing longitudinally

where

$$\frac{1}{f_{\parallel}} = \frac{qV_0}{p_0c}k\cos\phi_0 = \frac{qV_0}{p_0c}\frac{\pi}{L}\cos\phi_0 \quad (101)$$

Appendix A. A property of transfer matrices

Consider a linear second-order differential equation in the form of Hill's equation with a first derivative term included (to be general)

$$x'' + f(s)x' + g(s)x = 0 \qquad \text{(A1)}$$

Let's the two linearly independent solutions be x_1 and x_2. Then the Wronskian is

$$W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$$

Taking a derivative with respect to the independent variable s, we have

$$\frac{dW}{ds} = \begin{vmatrix} x_1' & x_2' \\ x_1' & x_2' \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ x_1'' & x_2'' \end{vmatrix} = -x_1(fx_2' + gx_2) + x_2(fx_1' + gx_1) = -fW$$

Integrating over ds, we get the solution for Wronskian

$$W(s) = W_0 e^{-\int_{s_0}^{s} f(s)ds}$$

where W_0 is a constant which is the Wronskian when $s = s_0$, i.e. $W_0 = W(0)$.

From this we see that if $f(s) = 0$, then $W = W_0 =$ constant

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Next, the solutions to Eq. (A1) can be written in the form

$$x_{1,2}(s) = m_{11}x_{1,2}(0) + m_{12}x'_{1,2}(0)$$
$$x'_{1,2}(s) = m_{21}x_{1,2}(0) + m_{22}x'_{1,2}(0) \qquad \text{(A2)}$$

This can be expressed in the form of a matrix equation:

$$\begin{pmatrix} x_1(s) & x_2(s) \\ x'_1(s) & x'_2(s) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_1(0) & x_2(0) \\ x'_1(0) & x'_2(0) \end{pmatrix} \qquad \text{(A3)}$$

Taking determinants on both sides, we find

$$W(s) = \det(M)\,W(0) \quad \text{where} \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad \text{and} \quad W(0) = W(s=0)$$

Since $W(s) = W(0)$ when $f(s) = 0$, we get

$$\det(M) = 1$$

We therefore proved that in the absence of a first derivative term, the determinant of the transfer matrix associated with the Hill's equation is always one. This also holds for a general 6×6 transfer matrix.

Formal (mathematically rigorous) proof for the determinant of symplectic matrices being equal to +1 is beyond the scope of this Lecture and can be found elsewhere [e.g. D. Rim, Adv. in Dynam. Sys. and App. (ADSA), 12, 1, (2017) 15 - 20].

Appendix B. Symplectic condition

Consider a column vector consisting of phase space coordinates.

$$\eta = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \qquad \text{(B1)}$$

Introducing a 6×6 symplectic matrix defined as

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \qquad \text{(B2)}$$

block-diagonal

then we can write the Hamilton's equations in the following form

$$\dot{\eta} = S\frac{\partial H}{\partial \eta} = S\nabla_\eta H \qquad \text{(B3)}$$

$$\dot{\eta} = \frac{d\eta}{dt}$$

e.g. In 2D,

$$\begin{pmatrix} \dot{x} \\ \dot{p}_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial p_x} \end{pmatrix} \qquad \text{(B4)}$$

Let's now introduce a general canonical transformation ζ that does not include time explicitly. General transformation can be expressed by $\zeta = \zeta(\eta)$. The Jacobian matrix of transformation ζ is

$$M_{ij} = \frac{\partial(Q,P)}{\partial(q,p)} = \frac{\partial \zeta_i}{\partial \eta_j}$$

$$\zeta = \begin{pmatrix} X \\ P_x \\ Y \\ P_y \\ Z \\ \Delta \end{pmatrix}$$

After the transformation, the equation of motion takes the form

$$\dot{\zeta} = M\dot{\eta}$$

So $$\dot{\eta} = S\frac{\partial H}{\partial \eta} = S\left(M^T \nabla_\zeta H\right) \qquad \left(\because\ \dot{\eta}_i = S_{ij}\frac{\partial H}{\partial \eta_j} = S_{ij}\frac{\partial \zeta_j}{\partial \eta_i}\frac{\partial H}{\partial \zeta_j} = S_{ij}M_{ji}\frac{\partial H}{\partial \zeta_j}\right)$$

and
$$\dot{\zeta} = M\dot{\eta} = MS\left(M^T \nabla_\zeta H\right) = MSM^T \nabla_\zeta H = MSM^T \frac{\partial H}{\partial \zeta}$$

If
$$MSM^T = S \qquad \text{(B5)}$$

the new canonical variables preserve the form of the Hamilton's equations: $\dot{\zeta} = S\frac{\partial H}{\partial \zeta}$

Eq. (B5) is called the symplectic condition. It also holds in the form of $M^T SM = S$

From this we have an important observation: If and only if Jacobian matrix M satisfies the symplectic condition, the canonical form using the same Hamiltonian is preserved. In that case, ζ and M are called the symplectic transformation and symplectic matrix respectively.

Symplectic condition is useful:

It provides a simple method to check accuracy of numerical tracking of particle motion.

It avoids accumulation of numerical errors in long term particle tracking.

It can be used to check the validity of transfer matrices.

Note that $\det(M) = 1$ is a necessary condition for symplectic matrix but not a sufficient condition. So the determinant of a symplectic matrix is always one, but not vice versa.

The symplectic condition is much stronger condition than $\det(M) = 1$.