Linear Optics

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1. Hamiltonians for accelerators (16:30 - 17:15, 8/4)
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5. Dispersion function and Longitudinal dynamics I (09:30 - 10:45, 8/7)
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1. Hamiltonians for Accelerators

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There are several possible approaches to study the motion of particles in classical mechanics; Newton-Lorentz equation, Lagrange equation, and Hamilton's equations, etc.

In accelerator physics, these three approaches are all used in one way or another.

Depending on the problem under consideration, one method is more convenient or powerful than the others.

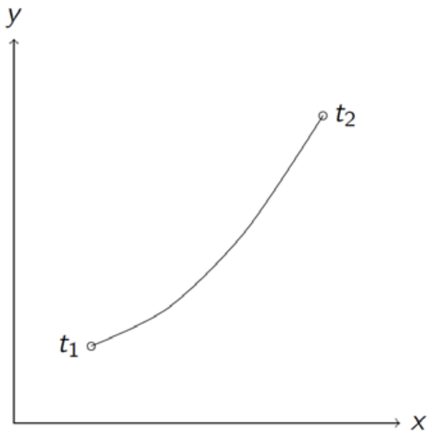
For nonlinear dynamics problem, Hamiltonian dynamics is in particular a power tool to get further insight of the behavior of particle's motion.

Even for linear beam dynamics, use of Hamiltonian can be advantageous as we can see soon.

In this Lecture, we shall adopt Hamiltonian method, which has an advantage of preserving the Liouville's theorem when changing variables to a new set of variables. This is particularly important when one investigates many-turn dynamics in circular particle accelerators such as synchrotron and storage ring.

Principle of least action

The action (or action integral) S is an integral of the function L along the trajectory



$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt \qquad (1)$$

where the function $L(q, \dot{q}, t)$ is a scalar quantity and called the Lagrangian, which is the key function in Lagrangian mechanics. Here q is called the generalized coordinate and $\dot{q}$ $(= dq/dt)$ the generalized velocity.

The principle of least action or Hamilton's principle holds that the system evolves in such a way that the action S is stationary. It can be shown that the Euler-Lagrange equation defines a path for which

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0 \qquad (2)$$

Euler-Lagrange equation

By taking variations on the Lagrangian one can easily derive the Euler-Lagrange equation (or Lagrange equation):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \qquad (3)$$

Here, $\frac{\partial L}{\partial \dot{q}}$ is called the generalized momentum: $\pi \equiv \frac{\partial L}{\partial \dot{q}}$ (4)

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In the case of a conservative force the Lagrangian is the difference of the kinetic energy $T(q,\dot{q})$ and potential energy $U(q)$:

$$L(q,\dot{q}) = T(q,\dot{q}) - U(q) \qquad (5)$$

And the generalized force is defined as

$$F = \frac{\partial L}{\partial q} \qquad (6)$$

Lack of uniqueness of the Lagrangian

Suppose for example there is a new Lagrangian $\tilde{L}(q,\dot{q},t)$ given by

$$\tilde{L}(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}G(q,t)$$

Then the action principle is

$$\tilde{S}(q(t)) = \int_{t_a}^{t_b} \tilde{L}(q,\dot{q})\,dt = S(q(t)) + G(q_b,t_b) - G(q_a,t_a)$$

$$\delta\tilde{S} = \delta S + \delta G(q_b,t_b) - \delta G(q_a,t_a)$$

But $\delta G = \frac{\partial G}{\partial q}\delta q + \frac{\partial G}{\partial t}\delta t = \frac{\partial G}{\partial q}\delta q \qquad \because\ \delta t = 0$

and the endpoints are fixed, $\delta q_a = \delta q_b = 0$, so $\delta G(q_b,t_b) = \delta G(q_a,t_a) = 0.$

Thus $\delta\tilde{S} = \delta S$. This means that L and $\tilde{L}$ result in the same equations of motion. Hence, the equations of motion are invariant under a shift of L by a total time derivative of a function of coordinates and time, $G(q,t)$. This is the basis of the canonical transformation and G is called the generating function.

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Lagrangian of a charged particle under electromagnetic fields

The relativistic Lagrangian of a charged particle under electromagnetic fields is given by (e.g. Jackson §12.1)

$$L = -\frac{mc^2}{\gamma} + q\boldsymbol{v}\cdot\boldsymbol{A}(\boldsymbol{r},t) - q\Phi(\boldsymbol{r},t) = -mc^2\sqrt{1-\left(\frac{\dot{\boldsymbol{r}}}{c}\right)^2} - q(\Phi - \dot{\boldsymbol{r}}\cdot\boldsymbol{A}) \qquad (7)$$

Φ : (electric) scalar potential, $\boldsymbol{A}$: vector potential, m : rest mass

The conjugate canonical momenta are

$$\boldsymbol{\pi}_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}_k} = \gamma m\dot{\boldsymbol{r}}_k + q\boldsymbol{A}_k = \boldsymbol{p}_k + q\boldsymbol{A}_k \qquad (k = 1,2,3,\cdots,n) \qquad (8)$$

$$\boldsymbol{p}_k = \gamma m\dot{\boldsymbol{r}}_k = \gamma\boldsymbol{\beta}_k mc \qquad \text{mechanical momenta}$$

n : number of generalized coordinates (i.e. number of degrees of freedom)

Substituting Eq. (7) into the Euler-Lagrange equation, Eq. (3), one can obtain the Lorentz equation as is shown in the below:

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \rightarrow 0 = \frac{d}{dt}\left(\frac{m\dot{\boldsymbol{r}}}{\sqrt{1-(\dot{\boldsymbol{r}}/\boldsymbol{c})^2}}\right) + q\left\{\frac{d\boldsymbol{A}}{dt} + \nabla(\Phi - \dot{\boldsymbol{r}}\cdot\boldsymbol{A})\right\}$$

$$\frac{d}{dt}\left(\frac{m\dot{\boldsymbol{r}}}{\sqrt{1-(\dot{\boldsymbol{r}}/\boldsymbol{c})^2}}\right) = \frac{d}{dt}(mc\gamma\boldsymbol{\beta}) = \boldsymbol{F} = -q\left\{\frac{d\boldsymbol{A}}{dt} + \nabla(\Phi - \dot{\boldsymbol{r}}\cdot\boldsymbol{A})\right\}$$

q is a charge not the generalized coord.

$$\boldsymbol{F} = -q\left\{\frac{d\boldsymbol{A}}{dt} + \nabla(\Phi - \dot{\boldsymbol{r}}\cdot\boldsymbol{A})\right\} \quad = -q\left\{\left(\frac{\partial\boldsymbol{A}}{\partial t} + (\dot{\boldsymbol{r}}\cdot\nabla)\boldsymbol{A}\right) + \nabla\Phi - \nabla(\dot{\boldsymbol{r}}\cdot\boldsymbol{A})\right\}$$

$$= -q\left\{\left(\frac{\partial\boldsymbol{A}}{\partial t} + (\dot{\boldsymbol{r}}\cdot\nabla)\boldsymbol{A}\right) + \nabla\Phi - \left((\dot{\boldsymbol{r}}\cdot\nabla)\boldsymbol{A} + \dot{\boldsymbol{r}}\times(\nabla\times\boldsymbol{A})\right)\right\}$$

$$= -q\left\{\frac{\partial\boldsymbol{A}}{\partial t} + \nabla\Phi - \dot{\boldsymbol{r}}\times(\nabla\times\boldsymbol{A})\right\} \quad = -q\left\{\nabla\Phi + \frac{\partial\boldsymbol{A}}{\partial t} - \dot{\boldsymbol{r}}\times(\nabla\times\boldsymbol{A})\right\}$$

With

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial\boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

we get the Lorentz equation as expected:

$$\boldsymbol{F} = q(\boldsymbol{E} + \dot{\boldsymbol{r}}\times\boldsymbol{B}) = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

Hamiltonian mechanics

The formulation of the laws of mechanics in terms of the Lagrangian, and of Lagrange's equations derived from it, presupposes that the mechanical state of a system is described by specifying its generalized co-ordinates q_k and generalized velocities $\dot{q}_k$. However, this is not the only possible way of description. A number of advantages, especially in the study of certain general problems of mechanics, attach to a description in terms of the generalized coordinates q_k and generalized momenta π_k of the system.

The passage from one set of independent variables to another can be affected by means of what is called in mathematics *Legendre's transformation.* In the present case this transformation is as follows.

The total differential of the Lagrangian as a function of generalized coordinates and generalized velocities is

$$dL(q_k, \dot{q}_k) = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k = \sum \dot{\pi}_k dq_k + \sum \pi_k d\dot{q}_k$$

Writing the last term on the r. h. s. in the form

$$\sum \pi_k d\dot{q}_k = d\left(\sum \pi_k \dot{q}_k\right) - \sum \dot{q}_k d\pi_k$$

we obtain

$$d\left(\sum \pi_k \dot{q}_k - L\right) = -\sum \dot{\pi}_k dq_k + \sum \dot{q}_k d\pi_k \qquad (9)$$

Hamiltonian and Hamilton's equations

Let's define the Hamiltonian:

$$H(\pi, q, t) = \sum_k \pi_k \dot{q}_k - L \quad (10)$$

$$= \boldsymbol{\pi} \cdot \boldsymbol{v} - L = \gamma mc^2 + q\Phi = c\sqrt{m^2c^2 + (\boldsymbol{\pi} - q\boldsymbol{A})^2} + q\Phi \quad [\because \text{Eqs. (7), (8)}]$$

or

$$H = \sqrt{m^2c^4 + p^2c^2} + q\Phi \quad \text{: total energy} \quad (11)$$

From the equation in differentials, Eq. (9)

$$dH = -\sum \dot{\pi}_k dq_k + \sum \dot{q}_k d\pi_k$$

we get Hamilton's equations :

$$\dot{q}_k = \frac{dq_k}{dt} = \frac{\partial H}{\partial \pi_k} \quad \text{and} \quad \dot{\pi}_k = \frac{d\pi_k}{dt} = -\frac{\partial H}{\partial q_k} \quad (12)$$

The total time derivative of the Hamiltonian is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum \frac{\partial H}{\partial q_k}\dot{q}_k + \sum \frac{\partial H}{\partial \pi_k}\dot{\pi}_k = \frac{\partial H}{\partial t}$$

where the last term on the r. h. s. follows by applying the Hamilton's equations. In particular, if the Hamiltonian does not depend explicitly on time, then $dH/dt = 0$, and we have the law of conservation of energy.

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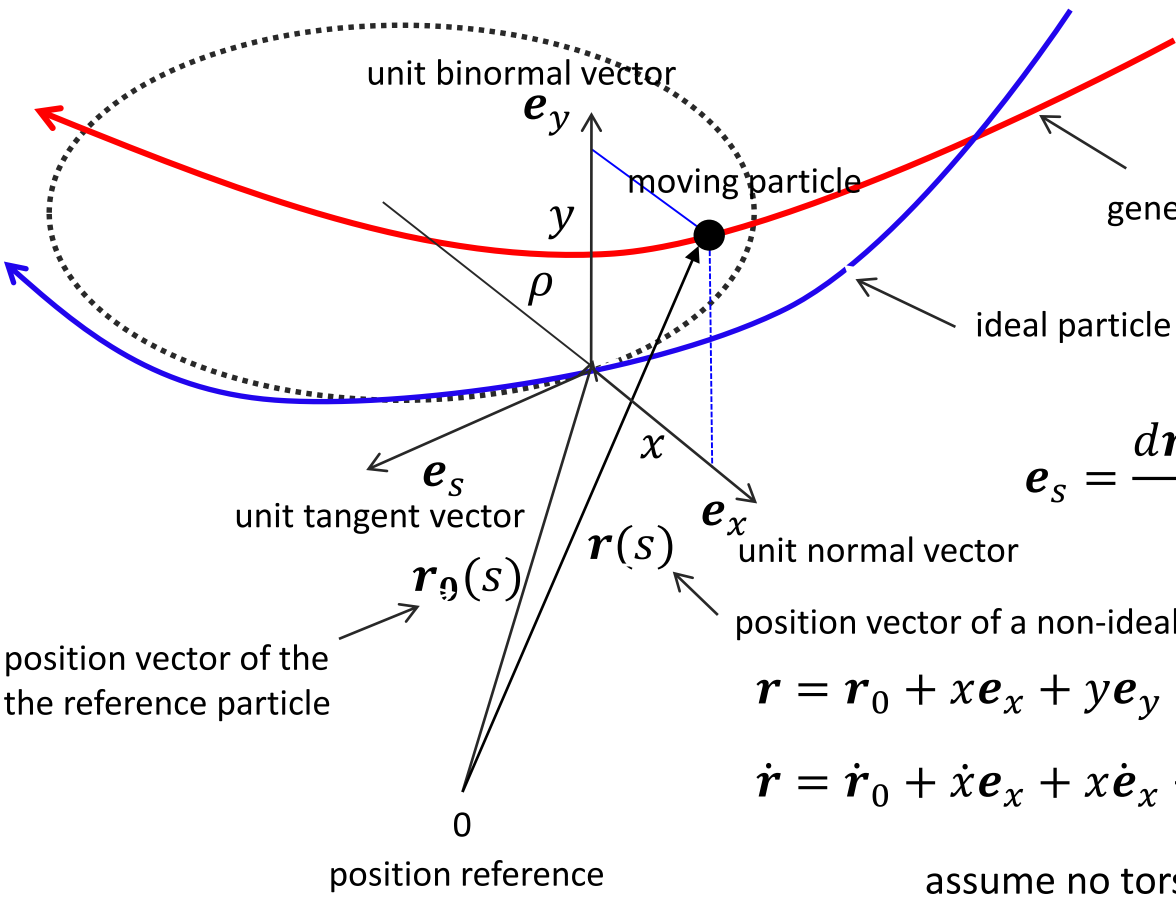
Curved coordinates (x, y, s) in circular accelerators

Let's now apply the Lagrangian and Hamiltonian to circular particle accelerators.

The coordinates (x, y, s) specifies a particle's position in right-handed coordinate system.

s is a coordinate along the reference particle's orbit (blue curve); it is in general curved.

We want to obtain equations of motion for a non-ideal particle (red curve).



$(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_s)$: unit vectors

$$\boldsymbol{e}_s = \frac{d\boldsymbol{r}_0(s)}{ds}$$: unit tangent vector

$$\boldsymbol{r} = \boldsymbol{r}_0 + x\boldsymbol{e}_x + y\boldsymbol{e}_y$$

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_0 + \dot{x}\boldsymbol{e}_x + x\dot{\boldsymbol{e}}_x + \dot{y}\boldsymbol{e}_y + y\dot{\boldsymbol{e}}_y \qquad \left(\cdot \equiv \frac{d}{dt}\right)$$

assume no torsion (planar orbit) then $\dot{\boldsymbol{e}}_y = 0$

Changing independent variable from t to s

Assuming that the longitudinal coordinate s is a monotonically increasing function of time t, we can change t by s.
This is for the convenience of accelerator beam treatment.
The new Lagrangian will be obtained through exchange of integral variable of the principle of least action from t to s :

$$0 = \delta \int L_0(\boldsymbol{r}, \dot{\boldsymbol{r}}) dt = \delta \int L_0\left(\boldsymbol{r}, \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt}\right)\frac{dt}{ds} ds = \delta \int L_0\left(\boldsymbol{r}, \frac{\boldsymbol{r}'}{t'}\right) t' ds \equiv \delta \int L\left(\boldsymbol{r}, \frac{\boldsymbol{r}'}{t'}\right) ds$$

where we have changed our previous Lagrangian to L_0 to indicate that its independent variable is time t. The new Lagrangian $L = L_0 t'$ is with independent variable s.

The new Lagrangian is then given by

$$L = L_0\left(\boldsymbol{r}, \frac{\boldsymbol{r}'}{t'}\right) t' = \left[-mc^2\sqrt{1 - \left(\frac{\boldsymbol{r}'}{ct'}\right)^2} - q\left(\Phi - \frac{\boldsymbol{r}'}{t'} \cdot \boldsymbol{A}\right)\right] t'$$

$$\because \dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \frac{\boldsymbol{r}'}{t'}$$

or

$$L(\boldsymbol{r}'; s) = -mc\sqrt{c^2 t'^2 - \boldsymbol{r}'^2} - q(\Phi t' - \boldsymbol{r}' \cdot \boldsymbol{A}) \qquad (13)$$

where $t' = \dfrac{dt}{ds}$ and $\boldsymbol{r}' = \dfrac{d\boldsymbol{r}}{ds}$

Curved coordinate system: Fresnet-Serret formula

We shall not prove the Fresnet-Serret formula (see e.g. Wikipedia) but just state the result.

The Frenet-Serret formula relates the derivatives of the tangent ($\boldsymbol{e}_s$), normal ($\boldsymbol{e}_x$), and binormal ($\boldsymbol{e}_y$) unit vectors, which are given in terms of each other:

$$\boldsymbol{e}_s = \frac{d}{ds}\boldsymbol{r}_0(s) = \boldsymbol{r}_0'(s)$$
$$\boldsymbol{e}_x' = -\tau\boldsymbol{e}_y + \kappa\boldsymbol{e}_s \qquad (14)$$
$$\boldsymbol{e}_y' = \tau\boldsymbol{e}_x$$
$$\boldsymbol{e}_s' = -\kappa\boldsymbol{e}_x$$

κ : curvature, inverse of radius

τ: torsion; zero for planar orbit as is the case for most of the existing accelerators

By applying the Frenet-Serret formula, s derivative of the position vector is expressed as

$$\boldsymbol{r}' = \boldsymbol{r}_0' + (x\boldsymbol{e}_x)' + (y\boldsymbol{e}_y)' = x'\boldsymbol{e}_x + y'\boldsymbol{e}_y + (1+\kappa x)\boldsymbol{e}_s$$

Note that for planar orbit, the torsion $\tau = 0$, and in this case we can easily derive Eq. (14) by considering the geometry, without resorting to the Fresnet-Serret formula.

Let's now change our notation to $h = \kappa$ (to follow the literature). Then

$$\frac{d\boldsymbol{r}}{ds} = \boldsymbol{r}' = \boldsymbol{r}_0' + (x\boldsymbol{e}_x)' + (y\boldsymbol{e}_y)' = x'\boldsymbol{e}_x + y'\boldsymbol{e}_y + (1+hx)\boldsymbol{e}_s \qquad (15)$$

$$h(s) = \frac{1}{\rho(s)}$$: local curvature of the orbit

Lagrangian in curved coordinate system with s independent variable

Let's substitute Eq. (15) into Eq. (13) to express the Lagrangian in terms of components:

$$L(\boldsymbol{r}'; s) = -mc\sqrt{c^2t'^2 - \boldsymbol{r}'^2} - q(\Phi t' - \boldsymbol{r}' \cdot \boldsymbol{A}) \qquad (A_s = \boldsymbol{A} \cdot \boldsymbol{e}_s)$$

$$= -mc\sqrt{c^2t'^2 - x'^2 - y'^2 - (1+hx)^2} + q\left(-\Phi t' + x'A_x + y'A_y + (1+hx)A_s\right) \qquad (16)$$

where we used $\boldsymbol{r}' \cdot \boldsymbol{A} = x'A_x + y'A_y + (1+hx)A_s$

To find the corresponding conjugate variables when the independent variable is s, we start with the action integral:

$$S = \int L_0(\boldsymbol{r}, \dot{\boldsymbol{r}})dt = \int \left(\sum \pi_k \dot{q}_k - H\right)dt = \int \left(\pi_x \dot{x} + \pi_y \dot{y} + \pi_s \dot{s} - H\right)dt$$

And the least-action principle is

$$\delta \int L_0(\boldsymbol{r}, \dot{\boldsymbol{r}})dt = \delta \int \left(\pi_x \dot{x} + \pi_y \dot{y} + \pi_s \dot{s} - H\right)dt = 0$$

With s the new independent variable, the least-action principle becomes

$$\delta \int L_0(\boldsymbol{r}, \dot{\boldsymbol{r}}) \frac{dt}{ds} ds = \delta \int \left[\pi_x x' + \pi_y y' + (-H)t' - (-\pi_s)\right]ds = 0$$

or

$$\delta \int L_0(\boldsymbol{r}, \dot{\boldsymbol{r}}) \frac{dt}{ds} ds = \delta \int \left[\pi_x x' + \pi_y y' + \pi_t t' - (-\pi_s)\right]ds = 0 \qquad (\pi_t = -H) \qquad (17)$$

Equation (17) tells us that with s being the new independent variable, the new canonically conjugate variables are (x, π_x), (y, π_y), $(t, \pi_t (= -H))$ and the corresponding new Hamiltonian is $(-\pi_s)$.

Canonical momenta in curved coordinates with s independent variable

With the new Lagrangian given in Eq. (16), we can obtain the canonical momenta:

$$\pi_x = \frac{\partial L}{\partial x'} = \frac{mcx'}{\sqrt{c^2 t'^2 - x'^2 - y'^2 - (1+hx)^2}} + qA_x$$

$$\pi_y = \frac{\partial L}{\partial y'} = \frac{mcy'}{\sqrt{c^2 t'^2 - x'^2 - y'^2 - (1+hx)^2}} + qA_y \qquad (18)$$

$$\pi_t = \frac{\partial L}{\partial t'} = \frac{-mc^3 t'}{\sqrt{c^2 t'^2 - x'^2 - y'^2 - (1+hx)^2}} - q\Phi$$

Note that the longitudinal canonical momentum π_t has the dimension of energy instead of momentum. It is because π_t is the conjugate momentum corresponding to the conjugate coordinate t, as we saw in Eq. (17).

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Hamiltonian with s an independent variable

The Hamiltonian in curved system with s an independent variable is expressed as

$$H_2 = -\pi_s = x'\pi_x + y'\pi_y + t'\pi_t - L \qquad\qquad A_s = \boldsymbol{A}\cdot\boldsymbol{e}_s$$

$$= -(1+hx)\sqrt{\left(\frac{\pi_t + q\Phi}{c}\right)^2 - (\pi_x - qA_x)^2 - \left(\pi_y - qA_y\right)^2 - m^2c^2} - q(1+hx)A_s \quad (19)$$

where Eqs. (16) and (18) were subtituted. Homework 1, prob. 2: Verify Eq. (19).

Canonical equations of motion are given by

$$x' = \frac{\partial H_2}{\partial \pi_x} \qquad \pi_x' = -\frac{\partial H_2}{\partial x}$$

$$y' = \frac{\partial H_2}{\partial \pi_y} \qquad \pi_y' = -\frac{\partial H_2}{\partial y} \qquad (20)$$

$$t' = \frac{\partial H_2}{\partial \pi_t} \qquad \pi_t' = -\frac{\partial H_2}{\partial t}$$

π_t is a constant of motion when Hamiltonian does not depend on time t explicitly.

Canonical momenta vs. kinetic momenta

Recall Eq. (18):

$$\pi_x = \frac{mcx'}{\sqrt{c^2t'^2 - x'^2 - y'^2 - (1+hx)^2}} + qA_x$$

$$= \frac{mcx'}{\sqrt{c^2t'^2 - \left(\frac{v}{\dot{s}}\right)^2}} + qA_x = \frac{m}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\frac{dx}{ds}\frac{ds}{dt} + qA_x = \gamma m\dot{x} + qA_x = p_x + qA_x$$

where we used Eq. (15), i.e.

$$x'^2 + y'^2 + (1+hx)^2 = r'^2 = \left(\frac{dr/dt}{ds/dt}\right)^2 = \left(\frac{v}{\dot{s}}\right)^2$$

And similarly we get

$$\pi_y = \gamma m\dot{y} + qA_y = p_y + qA_y$$

$$\pi_t = -\gamma mc^2 - q\Phi = -E \qquad (21)$$

So we see that the canonical momentum π_t is indeed equal to the negative of the total energy of a particle, as implied in Eq. (18) (from $-q\Phi$ term).

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Use of the canonical momenta is not convenient in practical calculation because they are very small quantities. Thus it is more convenient to scale the momenta.

The least action principle expressed by using the new Hamiltonian is

$$\delta \int \left(\frac{dx}{ds}\pi_x + \frac{dy}{ds}\pi_y + \frac{dt}{ds}\pi_t - H \right) ds = 0 \qquad (22)$$

where we have changed the notation such that H is now the new Hamiltonian, $-\pi_s$.

Let's divide by a constant p_0 which is usually taken to be the kinetic momentum of the reference (or central) particle in the distribution of particles

$$\delta \int \left(\frac{dx}{ds}\frac{\pi_x}{p_0} + \frac{dy}{ds}\frac{\pi_y}{p_0} + \frac{dt}{ds}\frac{\pi_t}{p_0} - \frac{H}{p_0} \right) ds = 0 \qquad (23)$$

Let's introduce the scaled canonical momenta and scaled Hamiltonian:

$$P_x = \frac{\pi_x}{p_0} \qquad P_y = \frac{\pi_y}{p_0} \qquad P_t = \frac{\pi_t}{p_0} \qquad H_1 = \frac{H}{p_0} \quad \text{: (scaled) Hamiltonian} \qquad (24)$$

New scaled Hamiltonian is from Eq. (19)

$$H_1 = \frac{H_2}{p_0}$$
$$= -(1+hx)\sqrt{\left(\frac{P_t + \frac{q\Phi}{p_0}}{c}\right)^2 - \left(P_x - \frac{qA_x}{p_0}\right)^2 - \left(P_y - \frac{qA_y}{p_0}\right)^2 - \frac{m^2c^2}{p_0^2}} - (1+hx)\frac{qA_s}{p_0} \qquad (25)$$

or

$$H_1 = -(1+hx)\sqrt{\left(\frac{P_t + \frac{q\Phi}{p_0}}{c}\right)^2 - \left(P_x - \frac{qA_x}{p_0}\right)^2 - \left(P_y - \frac{qA_y}{p_0}\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - (1+hx)\frac{qA_s}{p_0} \tag{26}$$

Note that canonical equations of motion do not change if one scales the Hamiltonian and canonical momenta by the same factor. Note also that this is not rigorously a canonical transformation but just a change of units.

Note also the following with the total energy $E = \sqrt{p^2c^2 + m^2c^4} + q\Phi$

$$\frac{p}{p_0} = \sqrt{\frac{(E - q\Phi)^2 - m^2c^4}{p_0^2c^2}} \equiv 1 + \delta \tag{27}$$

where $\delta = \frac{p - p_0}{p_0} = \frac{\Delta p}{p_0}$ is the relative momentum deviation.

Then Eq. (26) can also be written in the following form:

$$H_1 = -(1+hx)\sqrt{(1+\delta)^2 - \left(P_x - \frac{qA_x}{p_0}\right)^2 - \left(P_y - \frac{qA_y}{p_0}\right)^2} - (1+hx)\frac{qA_s}{p_0} \tag{28}$$

Introducing the scaled kinetic momenta and scaled vector potential as

$$q_x = P_x - \frac{qA_x}{p_0} = \frac{\pi_x - qA_x}{p_0} = \frac{p_x}{p_0}, \qquad q_y = P_y - \frac{qA_y}{p_0} = \frac{\pi_y - qA_y}{p_0} = \frac{p_y}{p_0}$$

$$a_s = \frac{qA_s}{p_0}$$

Note: q_x and q_y are not generalized coordinates.

Then Eq. (28) can be written in more compact form:

$$H_1 = -(1 + hx)\sqrt{(1 + \delta\,)^2 - q_x^2 - q_y^2} - (1 + hx)a_s \qquad (29)$$

The Hamiltonian H_1 given in Eq. (29) or Eq. (28) or Eq. (26) is expressed in terms of the canonical variables $\left(x, \frac{\pi_x}{p_0}\right), \left(y, \frac{\pi_y}{p_0}\right)$ and $\left(t, \frac{-E}{p_0}\right)$ where t is the (ever increasing) time and E is the total energy of the particle.

A more useful set of variables for beam dynamics replaces time and energy with their increments from the time and energy of the reference particle. And to make the energy a positive quantity, we now interchange the signs of the longitudinal variables, i.e. $\left(-t, \frac{E}{p_0}\right)$. This is allowed because by doing so the Hamilton's eqs. would not change.

So we first transform to the new time coordinate $\tau = -(t - t_0(s))$, where $t_0(s)$ is the time of the progress of the reference particle along the reference orbit, t is the time at which the observed particle passes the position s. Obviously the conjugate momentum (an energy) is $p_\tau = \frac{H - H_0}{p_0} = \frac{E - E_0}{p_0}$ where E_0 is the energy of the reference particle. Both $t_0(s)$ and $H_0(s)$ are given a priori. The time-difference variable measures the lead or lag in the arrival at s of the particle in question relative to that of the reference particle.
To summarize, the old variables (q, p) are $(-t, H)$ and the new variables (Q, P) are (τ, p_τ). To perform the change of variables in a formal way, let's use a generating function of the third type (i. e., function of new coordinates $Q = \tau$ and old momenta H). Apart from the identity transformation in the transverse plane, we choose

$$G_3(H, \tau; s) = -\frac{H - H_0(s)}{p_0}\tau + \frac{H}{p_0}t_0(s) \qquad (30)$$

The transformation equations are

$$-t = -\frac{\partial G_3}{\partial\left(\frac{H}{p_0}\right)} = -(t_0(s) - \tau), \qquad p_\tau = -\frac{\partial G_3}{\partial \tau} = \frac{H - H_0(s)}{p_0} = \frac{E - E_0(s)}{p_0}$$

$$\frac{\partial G_3}{\partial s} = \frac{E - E_0(s)}{p_0}\frac{dt_0}{ds} = \frac{E - E_0(s)}{p_0 v_0} \qquad (31)$$

where we have assumed E_0 is constant.

Then the new Hamiltonian becomes

$$H_2 = H_1 + \frac{\partial G_3}{\partial s} = -(1+hx)\sqrt{(1+\delta)^2 - q_x^2 - q_y^2} - (1+hx)a_s + \frac{E - E_0}{p_0 v_0} \quad (32)$$

or equivalently

$$H_2 = -(1+hx)\sqrt{(1+\delta)^2 - \left(\frac{\pi_x - qA_x}{p_0}\right)^2 - \left(\frac{\pi_y - qA_y}{p_0}\right)^2} - (1+hx)\frac{qA_s}{p_0} + \frac{E - E_0}{p_0 v_0} \quad (33)$$

Sometimes it is more convenient to work with the distance instead of time. Computer code like MAD-X uses $(-c\Delta t, p_t)$ where

$$-c\Delta t = -c\big(t - t_0(s)\big) = \frac{s}{\beta_0} - ct,$$
$$p_t = \frac{E - E_0}{p_0 c} = \frac{p_\tau}{c} = \frac{E(s)}{p_0 c} - \frac{1}{\beta_0} = \frac{1}{\beta_0}\frac{E - E_0}{E_0} = \frac{1}{\beta_0}\left(\frac{\gamma}{\gamma_0} - 1\right) \quad (34)$$

This is just identical to (τ, p_τ) except that the longitudinal coordinate is the distance instead of time. Then the Hamiltonian remains the same as Eq. (33).

The Hamiltonian given in Eq. (33) can be expressed in different form:

$$H_2 = \frac{p_t}{\beta_0} - (1+hx)\sqrt{\left(p_t + \frac{1}{\beta_0}\right)^2 - \left(\frac{\pi_x - qA_x}{p_0}\right)^2 - \left(\frac{\pi_y - qA_y}{p_0}\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - (1+hx)\frac{qA_s}{p_0} \quad (35)$$

Eq. (35) is our final Hamiltonian. This is identical with the Hamiltonian derived in Wolski's book, except the notation for energy deviation. Note that the origin of this Hamiltonian is MAD code (although not exactly the same) and we are following the MAD notation for energy deviation, i.e. p_t. Our Hamiltonian is an exact Hamiltonian derived in planar curvilinear coordinate system. Similar but different longitudinal canonical variables are used in BMAD, SAD, SixTrack, AT and Elegant codes as listed in the below. But they all become identical for ultra-relativistic particle, as in electron storage rings.

$$-c\Delta t = -c(t - t_0) = \frac{s}{\beta_0} - ct \qquad \Delta s = s - \beta ct \qquad \sigma = s - \beta_0 ct$$

$$p_t = \frac{E - E_0}{c\beta_0 p_0} \qquad \delta = \frac{p - p_0}{p_0} \qquad p_\sigma = \frac{E - E_0}{c\beta_0 p_0}$$

$$\pi_x = \gamma m\dot{x} + qA_x, \qquad \pi_y = \gamma m\dot{y} + qA_y, \qquad \pi_s = (1 + hx)\gamma m\dot{s} + q(1 + hx)A_s,$$

$$P_x = \frac{\pi_x}{p_0} \qquad P_y = \frac{\pi_y}{p_0} \qquad P_s = \frac{\pi_s}{p_0}$$

$$H\left(x, P_x, y, P_y, \Delta s, \delta; s\right) = \delta - P_s$$

$$H\left(x, P_x, y, P_y, \sigma, p_\sigma; s\right) = p_\sigma - P_s$$

$$H\left(x, P_x, y, P_y, -c\Delta t, p_t; s\right) = \frac{p_t}{\beta_0} - P_s$$