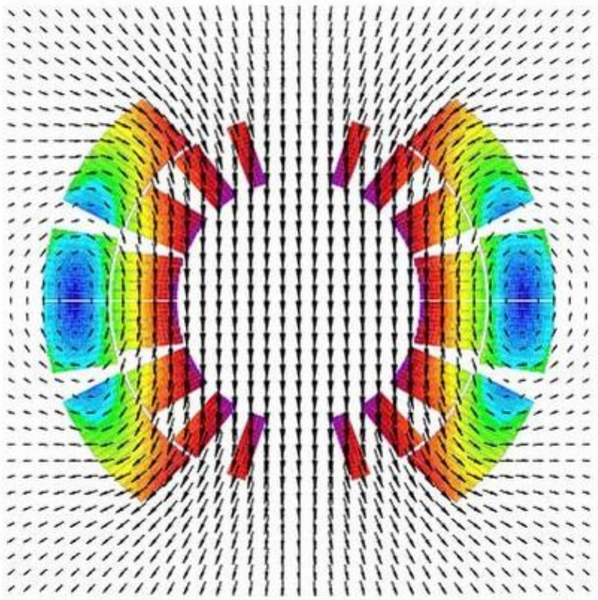
가속기 자석 Part-II

전자석 평가를 위한 해석 기초

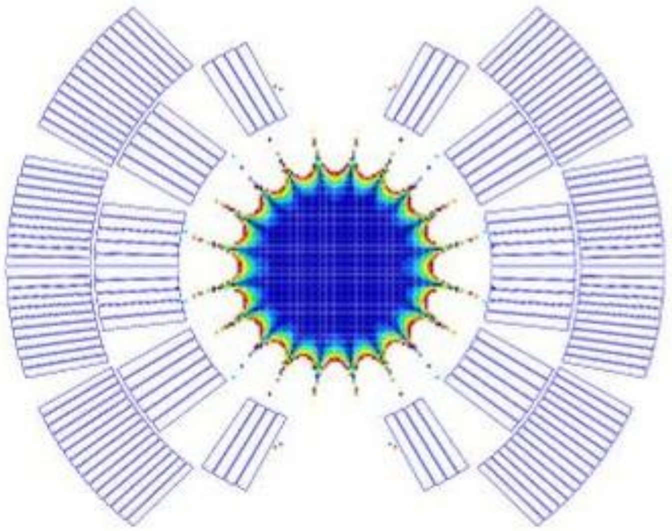
한가람
포항가속기연구소

2025-1 가속기실험실습
첨단원자력공학부/포항공과대학교

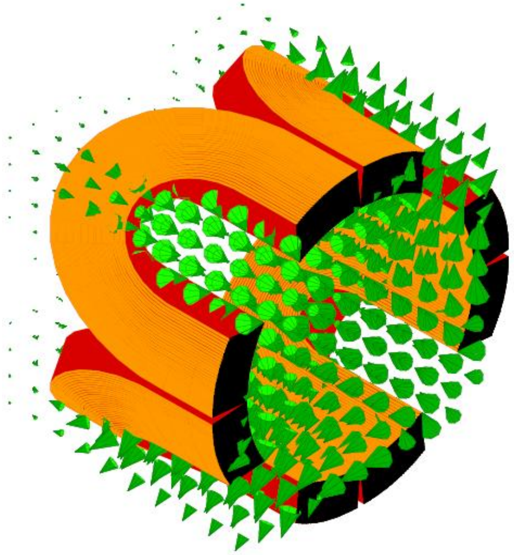
Field Quality



2D Field map



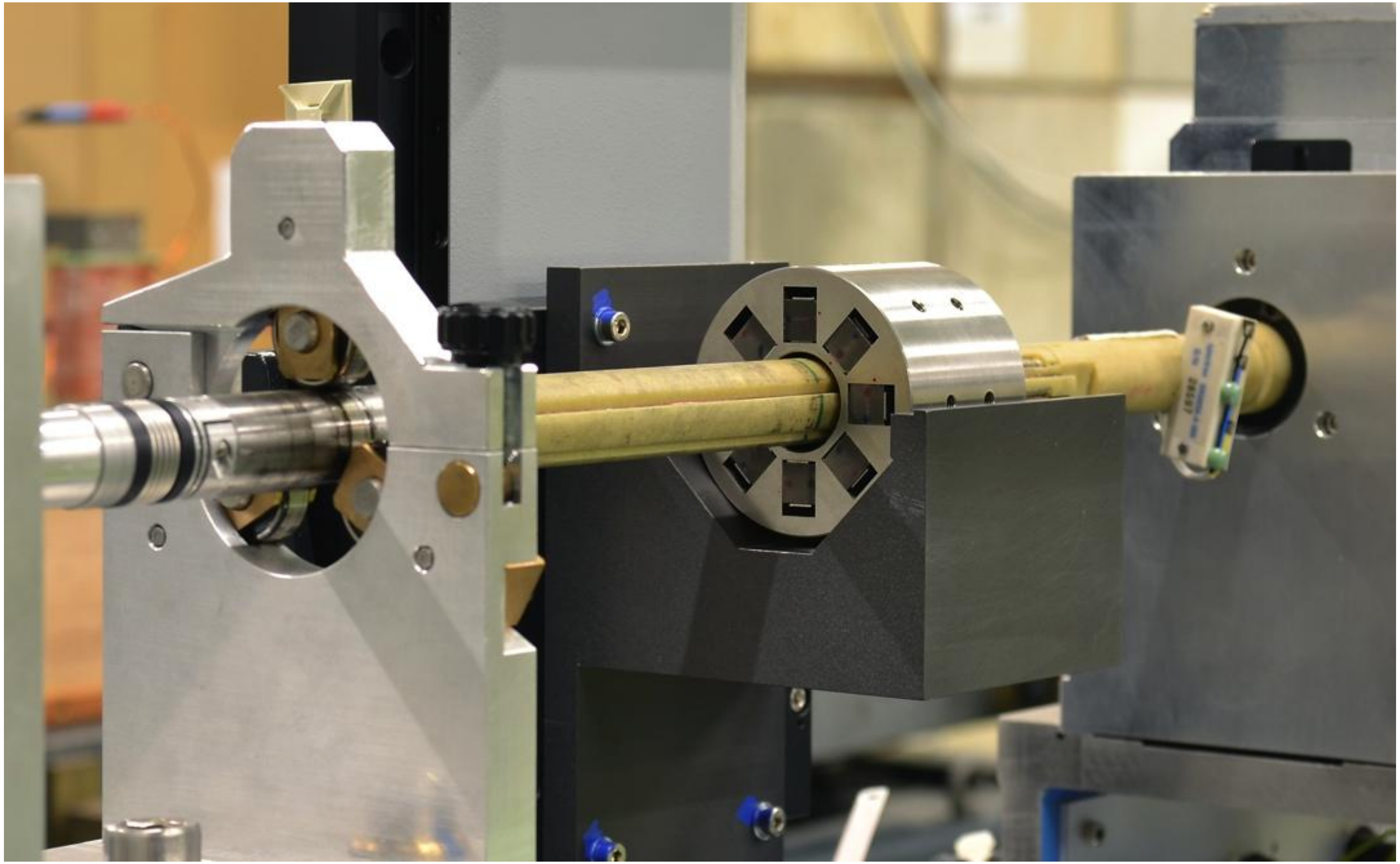
Good field region



3D Field map

출처: CERN S. Russenchucks 강의노트

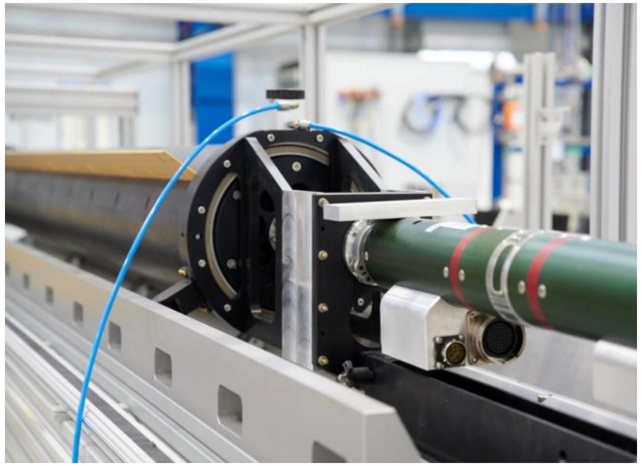
실제 자석은?



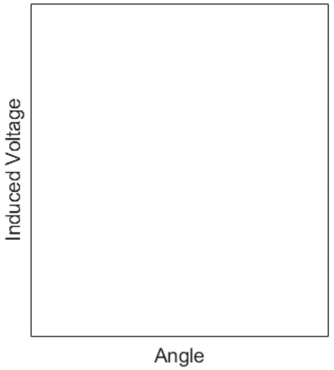
출처: CERN S. Russenchucks 강의노트

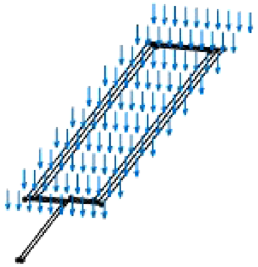
Rotating Coil Magnetometers

8 mm



380 mm





출처: CERN S. Russenchucks 강의노트

Review of Generalized Magnetic Multipole Model

(2023/10/24-)

$$\boxed{F = \vec{A} + iV}$$

- 2D magnetic field model in complex plane의 당위성 증명
- 전류가 있을 때와 없을 때 Maxwell equatio을 만족하는지 평가

$$\boxed{i\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y}}$$

- Cauchy Riemann 조건을 만족함을 보임
- 묵시적으로 <u>conformal mapping</u>이 가능함도 보임

$$\boxed{F = C_n z^n}$$

- 일반화된 멀티폴 포텐셜 모델링

$$\boxed{B_x = -\partial_x V, \qquad B_y = -\partial_x \vec{A}} \quad \boxed{B_x = \partial_y \vec{A}, \qquad B_y = -\partial_y V}$$

- 일반화된 멀티폴 필드 맵핑 확인

$$\boxed{B^* = i\partial_z F = iC_n(\partial_z z^n) = iC_n(nz^{n-1})}$$

- 일반화된 멀티폴 필드맵, 극좌표계에서 확인

$$B^*_{n,main}\Big|_{max} = \frac{1}{\pi}\oint\left(\vec{B}(\theta)\Big|_{r=r_0}\cdot\hat{r}\right)\sin n\theta\, d\theta \qquad C_{n,main} = -\frac{1}{nr_0^{n-1}}B^*_{n,main}\Big|_{max} \quad [\mathrm{T/m}^{n-1}]$$

$$B^*_{n,skew}\Big|_{max} = \frac{1}{\pi}\oint\left(\vec{B}(\theta)\Big|_{r=r_0}\cdot\hat{r}\right)\cos n\theta\, d\theta \qquad C_{n,skew} = -\frac{1}{nr_0^{n-1}}B^*_{n,skew}\Big|_{max} \quad [\mathrm{T/m}^{n-1}]$$

$$\boxed{\frac{|B^*_n|}{|B^*_N|} = \frac{n}{N}\left|\frac{C_n}{C_N}\right| r^{n-N}}$$

- Fundamental field component 에 정규화된 다극 성분 분석식

Modelling of 2D Magnetic Multipole : w/o current source

$$\boxed{F = \vec{A} + iV}$$

If there's no current source (inside the closed loop)

$$\vec{\nabla} \times \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Expressing $\vec{B}$ by potential functions $\vec{A}$ and V

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

$$\vec{B} = -\vec{\nabla} V = -\frac{\partial B}{\partial x}\hat{x} - \frac{\partial B}{\partial y}\hat{y} - \frac{\partial B}{\partial z}\hat{z}$$

Laplace equations, $\nabla^2 \vec{A} = 0$ and $\nabla^2 V = 0$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$
$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla^2 \vec{A} = 0 \quad (\because\ \vec{\nabla} \cdot \vec{A} = 0\ ) \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = -\vec{\nabla} \cdot \vec{\nabla} V$$

$$\nabla^2 V = 0 \tag{2}$$

$$\boxed{\therefore\ \nabla^2 F = \nabla^2 \vec{A} + i\nabla^2 V = 0}$$

$\boxed{F = \vec{A} + iV}$로 모델링 하면, current source가 없을 때 "Gauss's low for magnetism" of Maxwell's Eq.를 만족함

Modelling of 2D Magnetic Multipole : with current source

Ampere's circuital law with DC current source can be written as

$$\vec{\nabla} \times \vec{B} = (\partial_y B_z - \partial_z B_y)\hat{x} + (\partial_z B_x - \partial_x B_z)\hat{y} + (\partial_x B_y - \partial_y B_x)\hat{z}$$
$$= \mu\,(J_x\hat{x} + J_y\hat{y} + J_z\hat{z}) \qquad (1)$$

If B is given $B = B_x + iB_y$, then its conjugate form is

$$B^* = B_x - iB_y \quad \left(= i\frac{dF}{dz}\right)$$
$$= i\left(\frac{\partial F}{\partial x}\frac{dx}{dz} + \frac{\partial F}{\partial y}\frac{dy}{dz}\right) \qquad z = x + iy,$$
$$= i\left(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right) \qquad \frac{dz}{dy} = i, \qquad \frac{dz}{dx} = 1$$
$$= \frac{\partial F}{\partial y} - i\left(-\frac{\partial F}{\partial x}\right) \qquad (2)$$

Equating (1) and (2),

$$\frac{\partial}{\partial x}\left(-\frac{\partial F}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) = \mu J_z$$

$$\boxed{\nabla^2 F = \partial_x^2 F + \partial_y^2 F = -\mu J_z}$$

$\boxed{F = \vec{A} + iV}$ 로 모델링 하면, current source가 있을 때도 "Gauss's low for magnetism" of Maxwell's Eq.를 만족함

Modelling of 2D Magnetic Multipole : $F = \vec{A} + iV$

모델 특징,
코시-리만 기하
자장 설계법 연계

$$B^* = i\partial_z F = i\left(\frac{\partial A}{\partial x} + i\frac{\partial V}{\partial x}\right) = -\frac{\partial V}{\partial x} - i\left(-\frac{\partial A}{\partial x}\right)$$

$$\partial_z F = \frac{\partial}{\partial z}(A + iV) = \lim_{\Delta z \to 0}\frac{\Delta(A+iV)}{\Delta(x+iy)} = \lim_{\Delta z \to 0}\frac{\Delta(A+iV)/\Delta x}{1 + i\frac{\Delta y}{\Delta x}} = \frac{\partial A}{\partial x} + i\frac{\partial V}{\partial x}$$

$$\boxed{B_x = -\frac{\partial V}{\partial x}, \qquad B_y = -\frac{\partial \vec{A}}{\partial x}}$$

양변에 i 곱하면,

$$\boxed{i\left(\frac{\partial A}{\partial x} + i\frac{\partial V}{\partial x}\right) = \frac{\partial A}{\partial y} + i\frac{\partial V}{\partial y}}$$

$$B^* = i\partial_z F = i\left(-i\frac{\partial A}{\partial y} + \frac{\partial V}{\partial y}\right) = \frac{\partial A}{\partial y} - i\left(-\frac{\partial V}{\partial y}\right)$$

$$\partial_z F = \frac{\partial}{\partial z}(A + iV) = \lim_{\Delta z \to 0}\frac{\Delta(A+iV)}{\Delta(x+iy)} = \lim_{\Delta z \to 0}\frac{\Delta(A+iV)/\Delta y}{\frac{\Delta x}{\Delta y} + i} = -i\frac{\partial A}{\partial y} + \frac{\partial V}{\partial y}$$

$$\boxed{B_x = \frac{\partial \vec{A}}{\partial y}, \qquad B_y = -\frac{\partial V}{\partial y}}$$

$F = \vec{A} + iV$ satisfies Cauchy Riemann condition $i\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y}$, **which enables conformal mapping**

Multipoles : equipotential lines using F

2p, n=1 $F = C_1 z$
$= C_1(x + iy) = A + iV$
$$x = \frac{\vec{A}}{C_1}, \qquad y = \frac{V}{C_1}$$

4p, n=2 $F = C_2 z^2$
$$x^2 - y^2 = \frac{\vec{A}}{C_2}, \qquad 2xy = \frac{V}{C_2}$$

$_3C_1$

6p, n=3 $F = C_3 z^3$
$$x^3 - 3xy^2 = \frac{\vec{A}}{C_3}, \qquad 3x^2y - y^3 = \frac{V}{C_3}$$

$_4C_1 \quad {}_4C_2$

8p, n=4 $F = C_4 z^4$
$$x^4 - 6x^2y^2 + y^4 = \frac{\vec{A}}{C_4}, \qquad 4x^3y - 4xy^3 = \frac{V}{C_4}$$

$_5C_1 \quad {}_5C_2 \quad {}_5C_3$

10p, n=5 $F = C_5 z^5$
$$x^5 - 10x^3y^2 + 5xy^4 = \frac{\vec{A}}{C_5}, \qquad 5x^4y - 10x^2y^3 + y^5 = \frac{V}{C_4}$$

$\vdots \qquad \vdots \qquad \vdots$

Multipoles : Field strength using F

2p, n=1 $x = \frac{\vec{A}}{C_1}, \quad y = \frac{V}{C_1},$

$$B_x = -\partial_x V = 0, \quad B_y = -\partial_y V = -C_1$$

4p, n=2 $x^2 - y^2 = \frac{\vec{A}}{C_2}, \quad 2xy = \frac{V}{C_2},$

$$B_x = -\partial_x V = -2C_2 y, \quad B_y = -\partial_x A = -2C_2 x$$

6p, n=3 $x^3 - 3xy^2 = \frac{\vec{A}}{C_3}, \quad 3x^2y - y^3 = \frac{V}{C_3}$

$$B_x = \partial_y A = -6C_3 xy,$$
$$B_y = -\partial_y V = -3C_3x^2 + 3C_3y^2$$

$$\vdots$$

8p, n=4 $x^4 - 6x^2y^2 + y^4 = \frac{\vec{A}}{C_4}, \quad 4x^3y - 4xy^3 = \frac{V}{C_4}$

10p, n=5 $x^5 - 10x^3y^2 + 5xy^4 = \frac{\vec{A}}{C_5}, \quad 5x^4y - 10x^2y^3 + y^5 = \frac{V}{C_4}$

$$\vdots$$

편한 식 골라서...

$$\boxed{B_x = -\partial_x V, \quad B_y = -\partial_x \vec{A}}$$

$$\boxed{B_x = \partial_y \vec{A}, \quad B_y = -\partial_y V}$$

X-V-A, YA-V 암기용

Multipoles : field strength using F in polar coordinate

필드 맵핑
극좌표계

$$\boxed{B^* = i\partial_z F = iC_n(\partial_z z^n) = iC_n(nz^{n-1})}$$

2p, n=1 $B_1^* = iC_1$ $B_{1x} = 0,$ $B_{1y} = -C_1$

4p, n=2 $B_2^* = iC_2\left(2\,|z|\,e^{i\theta}\right) = i2C_2|z|(\cos\theta + i\sin\theta)$ $B_{2x} = -2C_2|z|\sin\theta\,,$ $B_{2y} = -2C_2|z|\cos\theta$

6p, n=3 $B_3^* = iC_3\left(3\,|z|^2\,e^{i2\theta}\right) = i3C_3|z|^2(\cos 2\theta + i\sin 2\theta)$ $B_{3x} = -3C_3|z|^2\sin 2\theta\,,$ $B_{3y} = -3C_3|z|^2\cos 2\theta$

8p, n=4 $B_4^* = iC_4\left(4\,|z|^3\,e^{i3\theta}\right) = i4C_4|z|^3(\cos 3\theta + i\sin 3\theta)$ $B_{4x} = -4C_4|z|^3\sin 3\theta\,,$ $B_{4y} = -4C_4|z|^3\cos 3\theta$

$\vdots$ $\vdots$

Multipole Analysis

$$\boxed{B^* = i\partial_z F = iC_n(\partial_z z^n) = iC_n(nz^{n-1})}$$

$$B_{nx} = -nC_n|z|^{n-1}\sin(n-1)\theta$$
$$B_{ny} = -nC_n|z|^{n-1}\cos(n-1)\theta$$

Tr plane에서 중심 기준 r 고정, 한 바퀴 적분했을 때 orthogonality에 의해 아래와 같이 적분식으로 정리 가능

$$C_n = \frac{-1}{n\pi r_0^{n-1}} \oint B_x(\theta)\Big|_{r=r_0} \cdot \sin(n-1)\theta\, d\theta \qquad [\mathrm{T/m^{n-1}}]$$

$$C_n = \frac{-1}{n\pi r_0^{n-1}} \oint B_y(\theta)\Big|_{r=r_0} \cdot \cos(n-1)\theta\, d\theta \qquad [\mathrm{T/m^{n-1}}]$$

그런데 이렇게 쓰면 dipole skew만 볼 수 없음, 아래와 같이 바꿔 쓸 수 있음

$$\boxed{C_{n,main} = \frac{-1}{n\pi r_0^{n-1}} \oint \left(\vec{B}(\theta)\Big|_{r=r_0} \cdot \hat{r}\right) \sin n\theta\, d\theta}$$

$$\boxed{C_{n,skew} = \frac{-1}{n\pi r_0^{n-1}} \oint \left(\vec{B}(\theta)\Big|_{r=r_0} \cdot \hat{r}\right) \cos n\theta\, d\theta}$$

Multipole Analysis: multipole field error

Normalized to the fundamental component $\frac{|B_n^*|}{|B_N^*|}$ is proportional to $\left|\frac{C_n}{C_N}\right|$

$$\boxed{\frac{|B_n^*|}{|B_N^*|} = \frac{n}{N} r_0^{n-N} \left|\frac{C_n}{C_N}\right|}$$

$$C_n' = C_n r_0^n \quad [\mathrm{T \cdot m}]$$

$$\left|\frac{C_n}{C_N}\right| = \left|\frac{C_n'}{C_N'}\right|$$

$$B_{n,main}^*\Big|_{max} = \frac{1}{\pi}\oint\left(\vec{B}(\theta)\Big|_{r=r_0}\cdot\hat{r}\right)\sin n\theta\, d\theta$$

$$B_{n,skew}^*\Big|_{max} = \frac{1}{\pi}\oint\left(\vec{B}(\theta)\Big|_{r=r_0}\cdot\hat{r}\right)\cos n\theta\, d\theta$$

$$C_{n,main}' = -\frac{r_0}{n} B_{n,main}^*\Big|_{max} \quad [\mathrm{T \cdot m}]$$

$$C_{n,skew}' = -\frac{r_0}{n} B_{n,skew}^*\Big|_{max} \quad [\mathrm{T \cdot m}]$$

$$C_{n,main} = -\frac{1}{n r_0^{n-1}} B_{n,main}^*\Big|_{max} \quad [\mathrm{T/m}^{n-1}]$$

$$C_{n,skew} = -\frac{1}{n r_0^{n-1}} B_{n,skew}^*\Big|_{max} \quad [\mathrm{T/m}^{n-1}]$$

Code Implementation

```
double_t Bmain, Bskew;
ThreeVector x_probe;
double_t poles[13] = {1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 11.0, 12.0, 13.0};
fo_multipole << time*beta_e(p.GetKineticEnergy())*c;
for (const auto& mp : poles)
{
    Bmain = 0.; Bskew = 0.;
    fo_multipole << ",";
    for (const auto& th : ths)
    {
        ThreeVector dr = mp_radius * x_pr_hat;
        dr.rotate(s_hat, th);
        x_probe = v0 + dr;
        ft->GetBField(x_probe, e3b3);
        Bmain += ThreeVector(e3b3[3], e3b3[4], e3b3[5]).dot(dr.unit()) * sin(mp * th) * dth;
        Bskew += ThreeVector(e3b3[3], e3b3[4], e3b3[5]).dot(dr.unit()) * cos(mp * th) * dth;
        fo_multipole_scanpath.write((char*)&x_probe, sizeof(ThreeVector));
    }
    Bmain /= pi; Bskew /= pi;
    double_t Cpr_main = -Bmain * mp_radius / mp, Cpr_skew = -Bskew * mp_radius / mp;
    double_t Cmain = -Bmain / mp / pow(mp_radius, mp-1), Cskew = -Bskew / mp / pow(mp_radius, mp - 1);
    fo_multipole
        << Bmain / tesla << "," << Bskew / tesla << ","
        << Cpr_main / (tesla * m) << "," << Cpr_main / (tesla * m) << ","
        << Cmain / (tesla / pow(m, mp - 1)) << "," << Cmain / (tesla / pow(m, mp - 1));
}
```

Artificial Tabulated Field-map Generation

1. Sector bend dipole

2. Sector bend dipole + quadrupole

1. Sector bend dipole

Specification

ROT_ANGLE = 6 * deg
C1 = 1.4 * tesla
BRHO = brho_e(4*GeV)
RHO = BRHO/C1

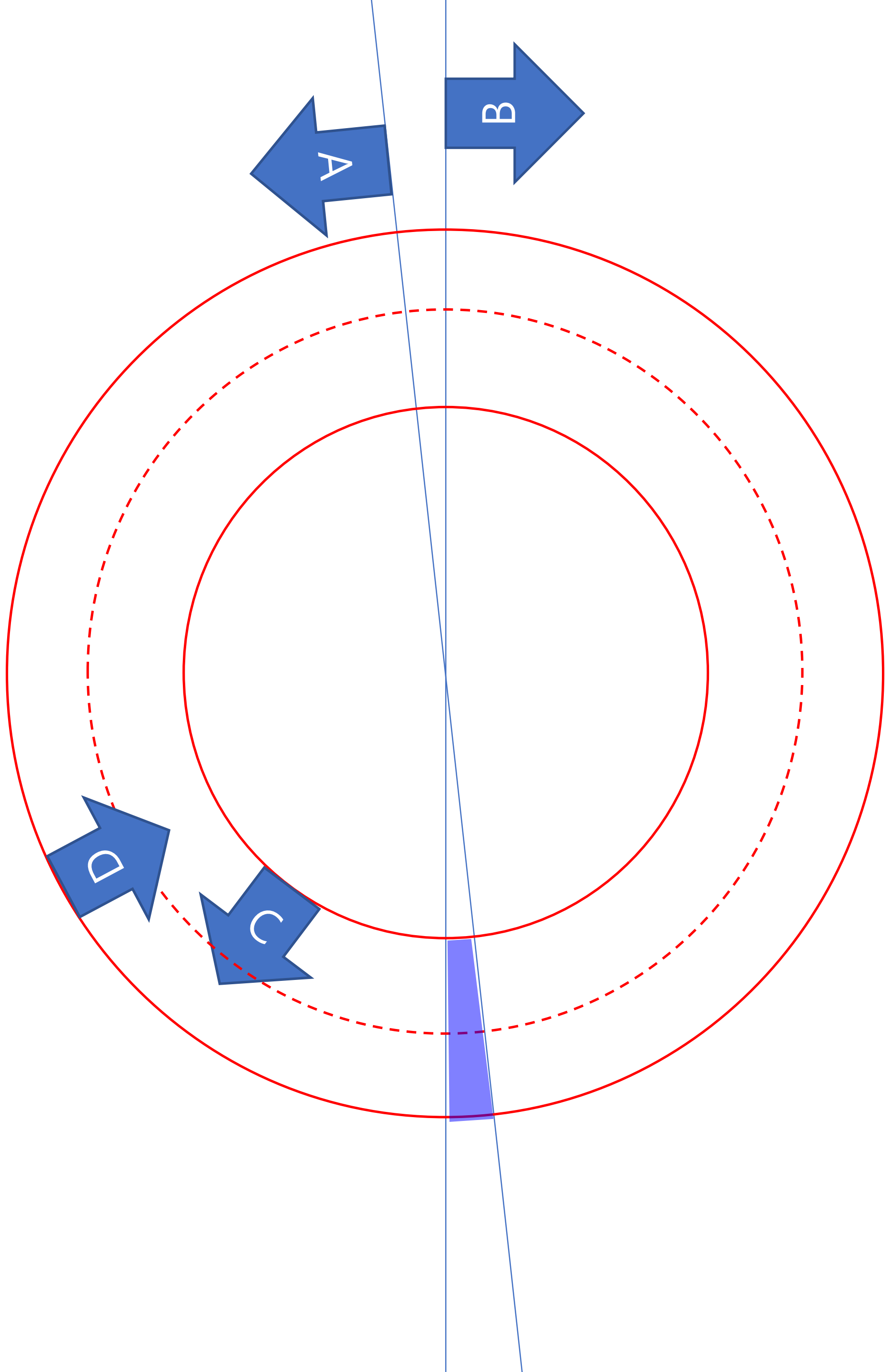
Geometry computation

$$\vec{x} = \{\vec{x}_i \mid A \,\&\, B \,\&\, C \,\&\, D \,\&\, E\}$$

$$\begin{aligned}
A&:\quad D_{l1}(\vec{x}_i) > 0,\\
B&:\quad D_{l2}(\vec{x}_i) < 0,\\
C&:\quad R_{inner}(\vec{x}_i) > \rho - W_0,\\
D&:\quad R_{outer}(\vec{x}_i) < \rho + W_0,\\
E&:\quad |\vec{x}_i \cdot \hat{y}| < G_0
\end{aligned}$$

$$B(\vec{x}) = \mathrm{C}_1,$$

otherwise $B(\vec{x}) = 0$



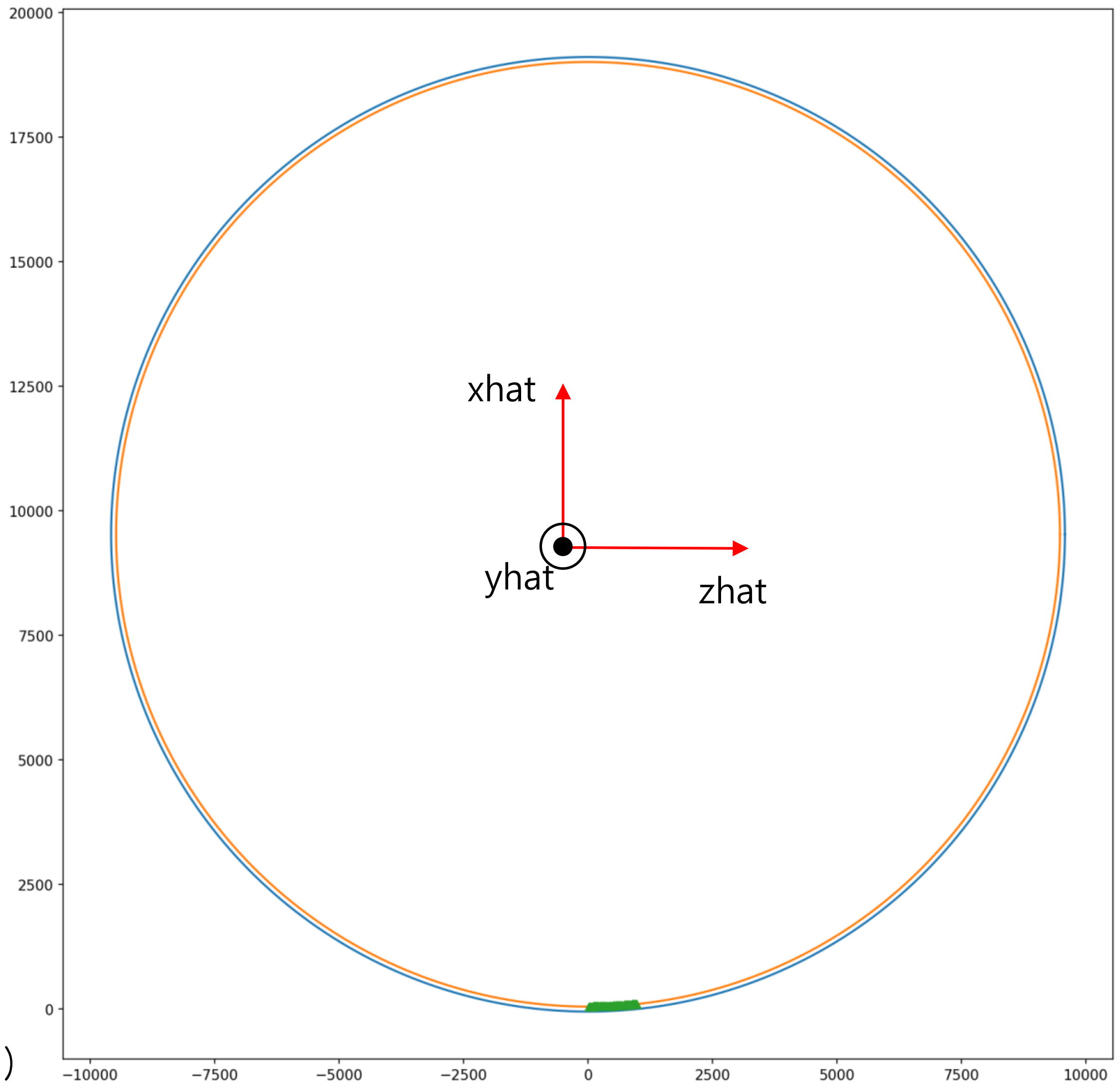
1. Sector bend dipole (cont'd)

Code

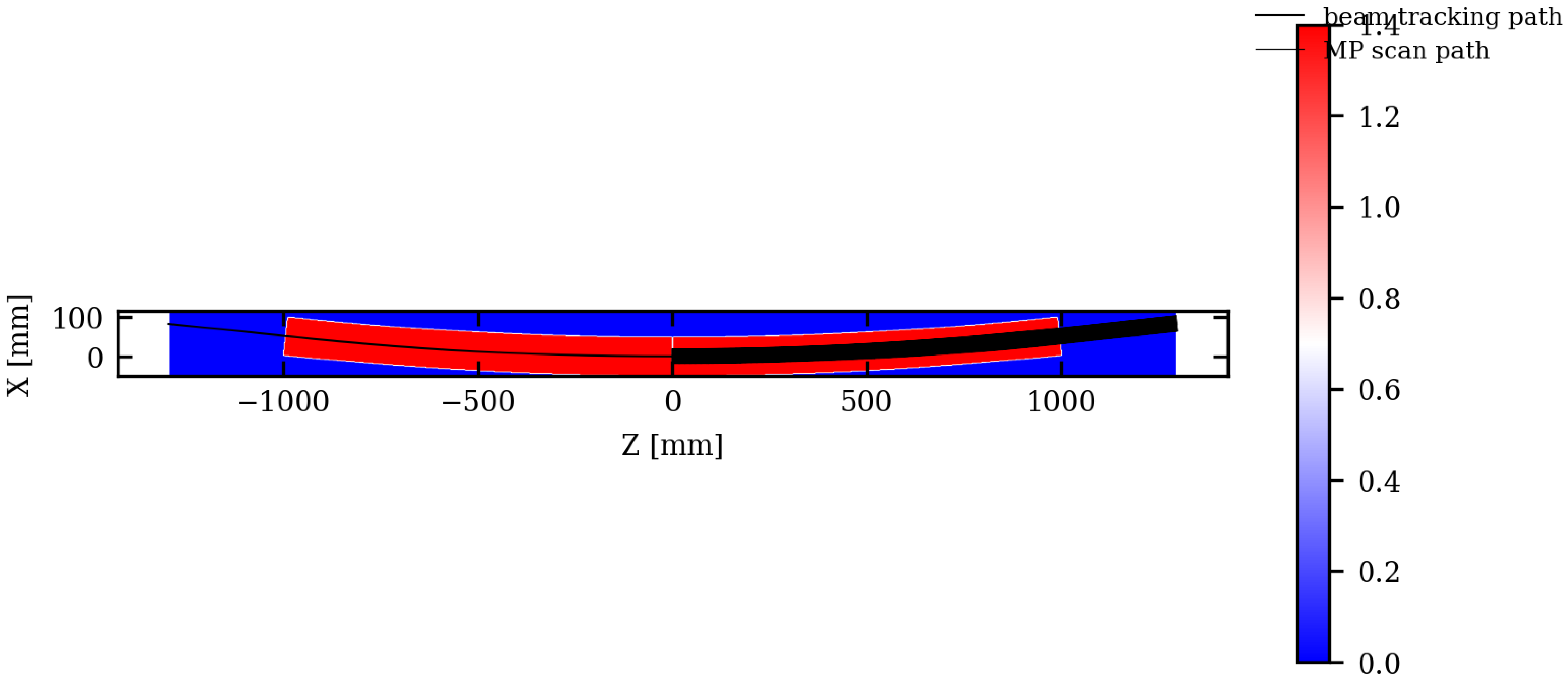
```
xtab0, xtab1 = -half_width, end_point.y
ytab0, ytab1 = 0, half_gap
ztab0, ztab1 = 0, end_point.x
dx = dy = dz = 1 * mm
Bx, By, Bz = 0, 0, 0
with open("dp_field_1mm.table","w") as fmap:
    z = ztab0
    while z < ztab1:
        y = ytab0
        while y < ytab1:
            x = xtab0
            while x < xtab1:
                v_i = TwoVector(z,x)
                a = line_1.get_distance_to_point(v_i)
                b = line_2.get_distance_to_point(v_i)
                if a < 0 and b > 0 and not circ_ir.is_inside(v_i) and circ_or.is_inside(v_i):
                    By = C1
                else:
                    By = 0
                fmap.write(f"{x/mm:.6f} {y/mm:.6f} {z/mm:.6f} "
                           +f"{Bx/tesla:.6f} {By/tesla:.6f} {Bz/tesla:.6f}₩n")
                x += dx
            y += dy
        z += dz
```

```
half_width = 5*cm
half_gap = 3*cm
z_margin = 5*cm
drift_length = 30*cm
```

```
ROT_ANGLE = 6 * deg
C1 = 1.4 * tesla
BRHO = brho_e(4*GeV)
RHO = BRHO/C1
```



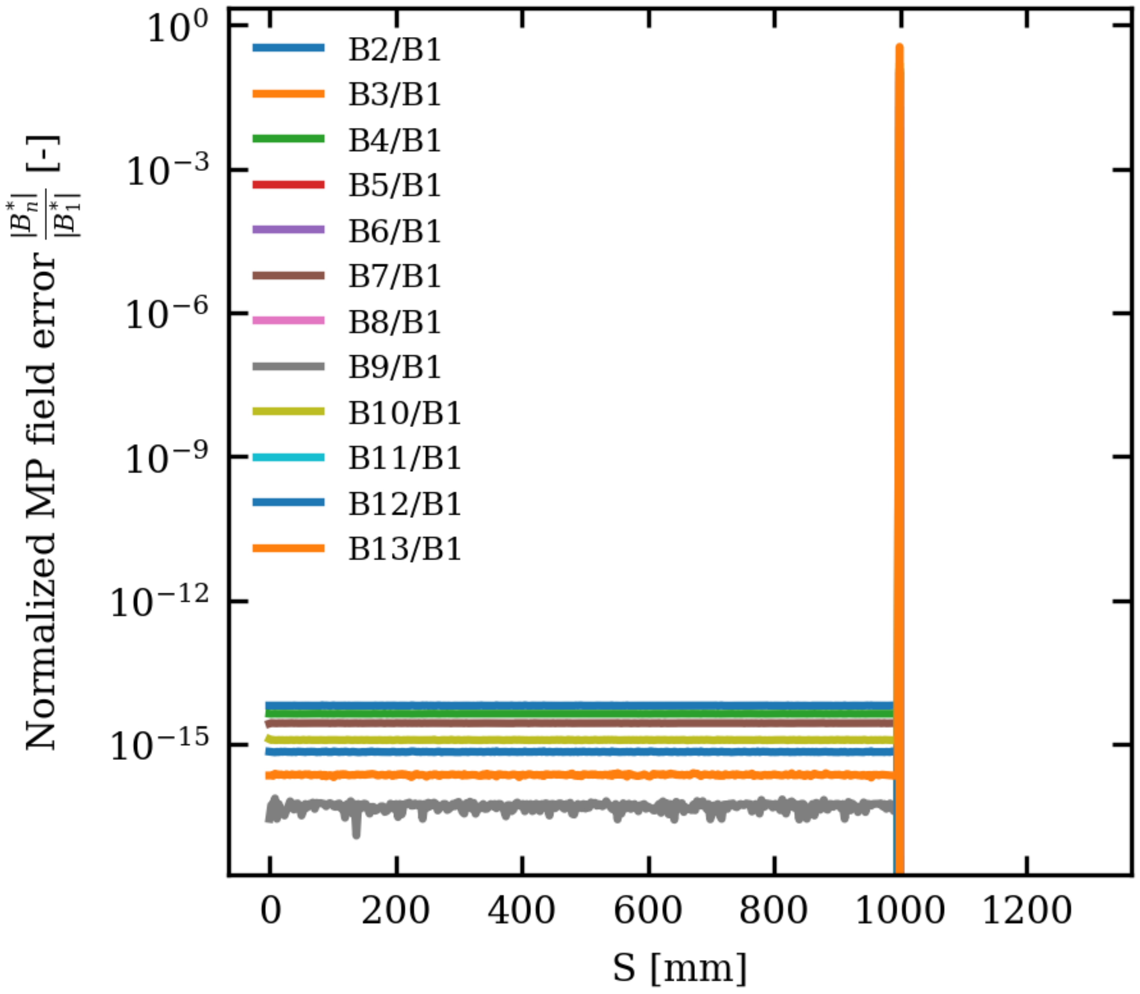
1. Sector bend dipole (cont'd), results

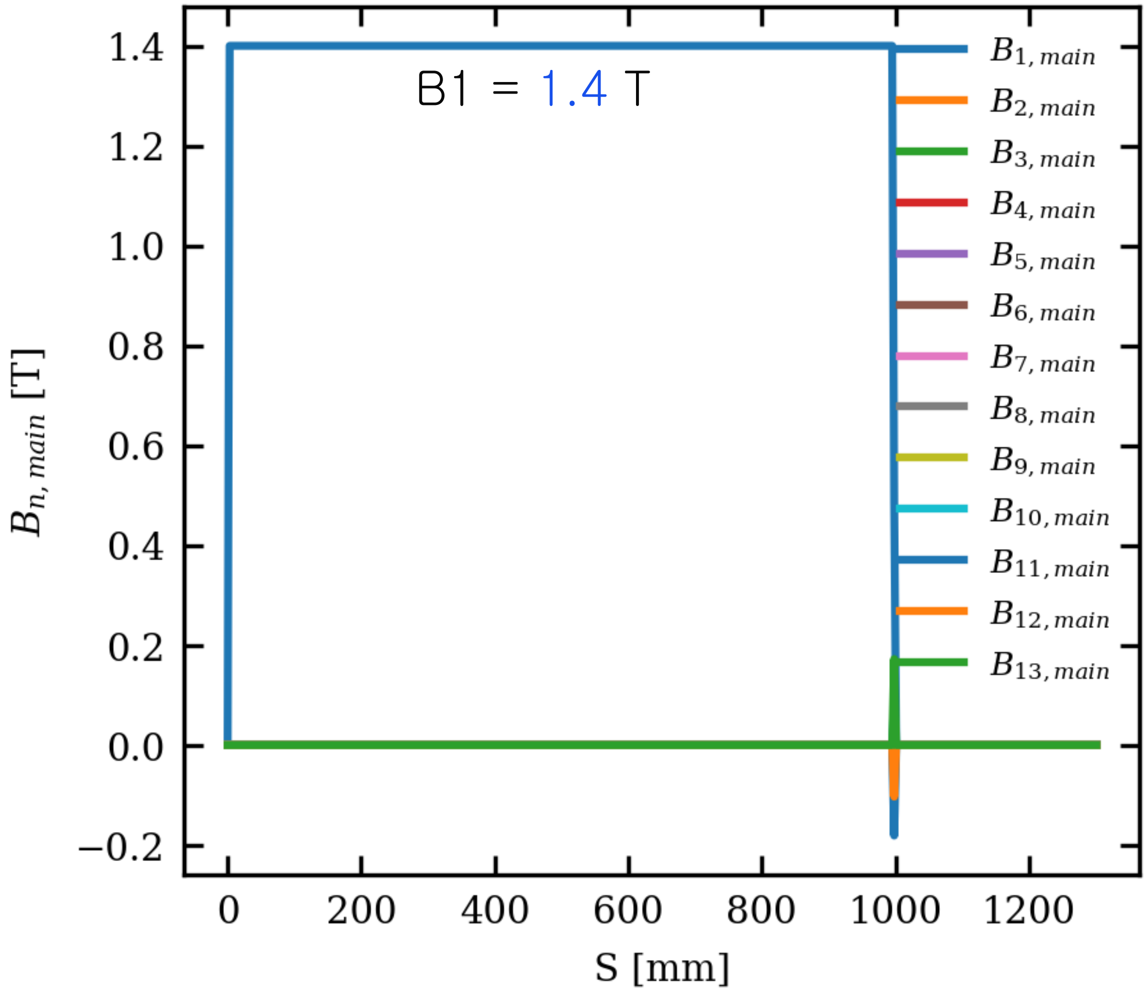


1. Sector bend dipole (cont'd), results

$$\frac{|B_n^*|}{|B_N^*|}$$

$$B^*_{n,main}\Big|_{max} = \frac{1}{\pi}\oint\left(\vec{B}(\theta)\Big|_{r=r_0}\cdot\hat{r}\right)\sin n\theta\, d\theta$$

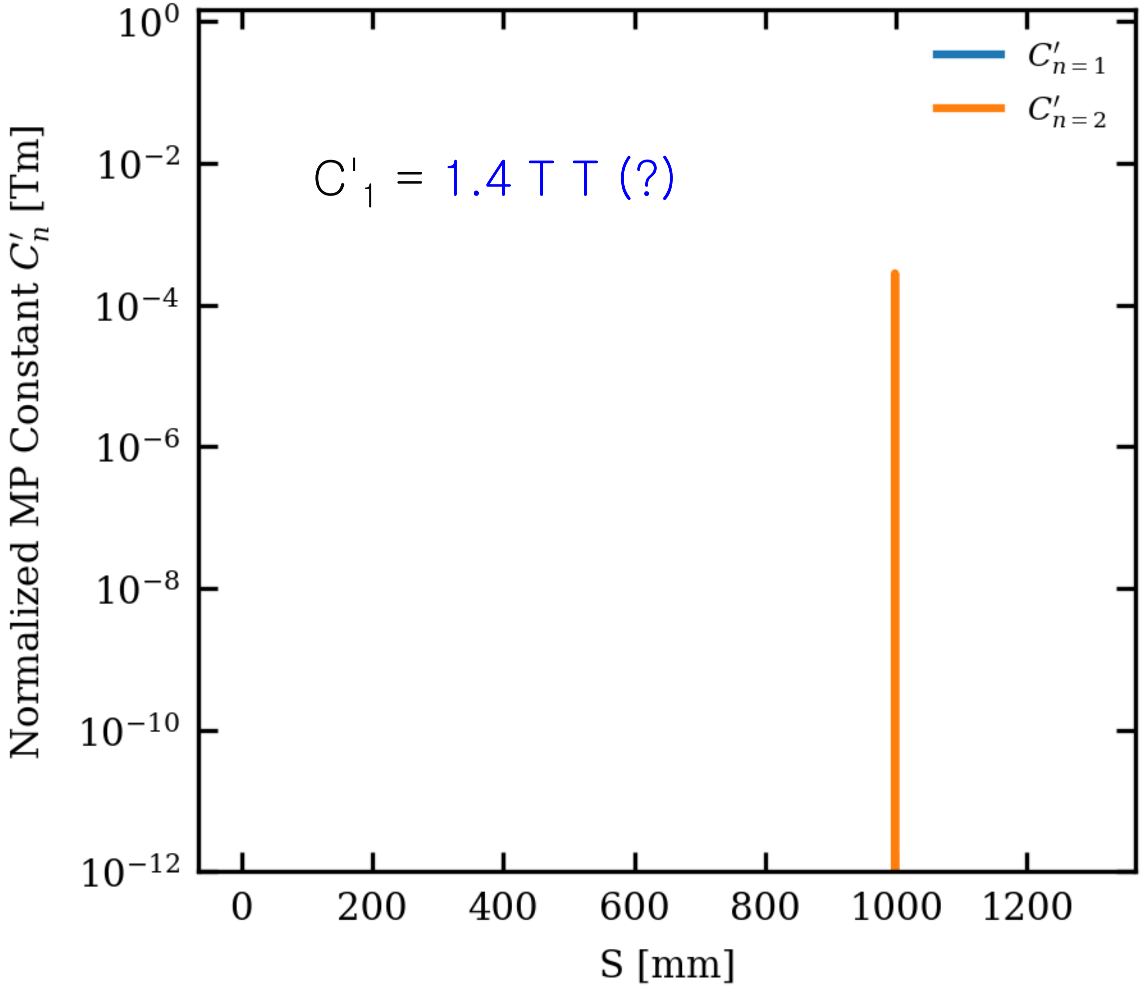


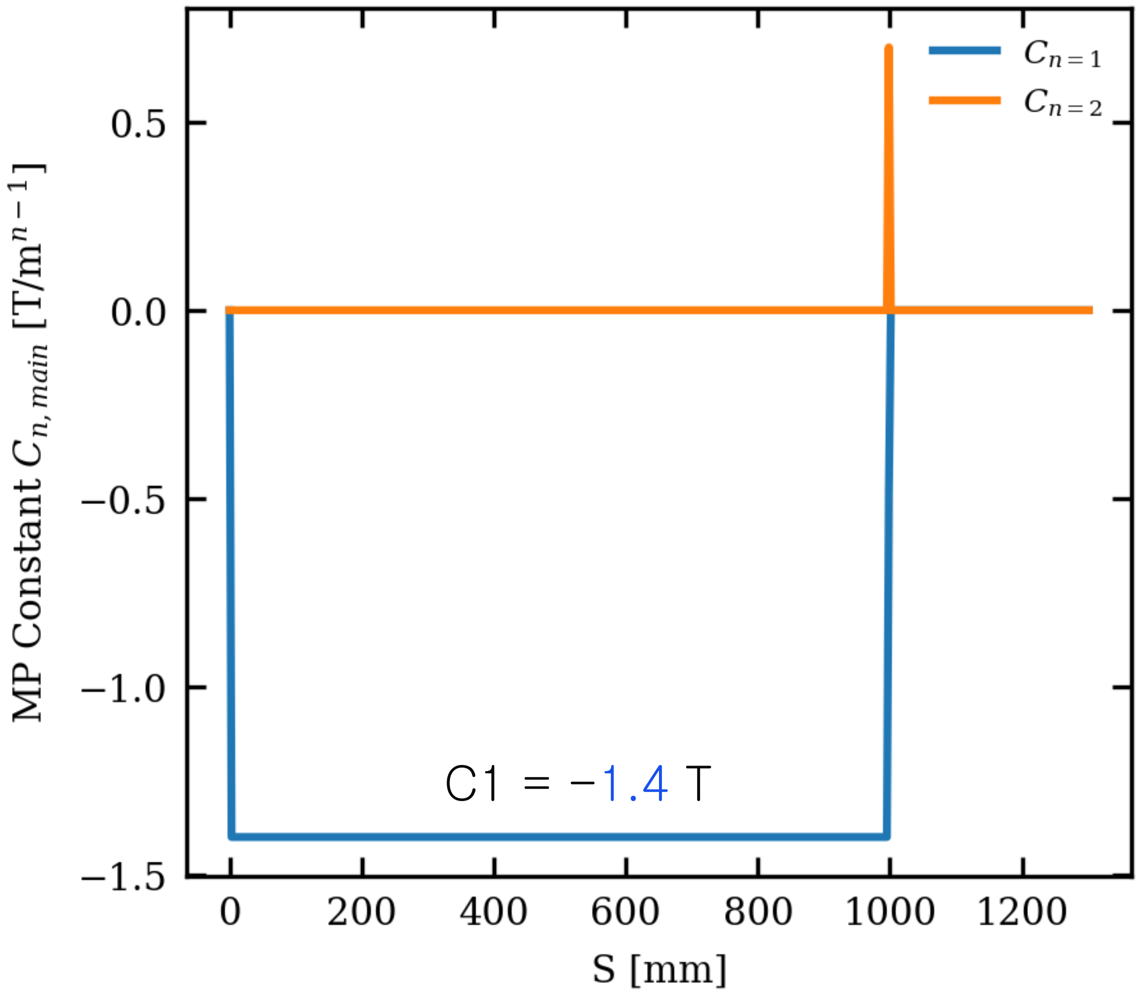


1. Sector bend dipole (cont'd), results

$$C'_{n,main} = -\frac{r_0}{n} B^*_{n,main}\Big|_{max} \quad [\mathrm{T\cdot m}]$$

$$C_{n,main} = -\frac{1}{n r_0^{n-1}} B^*_{n,main}\Big|_{max} \quad [\mathrm{T/m^{n-1}}]$$





2. Sector bend dipole + quadrupole

$$\vec{B}'(\vec{x}') = (-2C_2 y')\,\hat{x}' + (C_1 - 2C_2 x')\,\hat{y}'$$

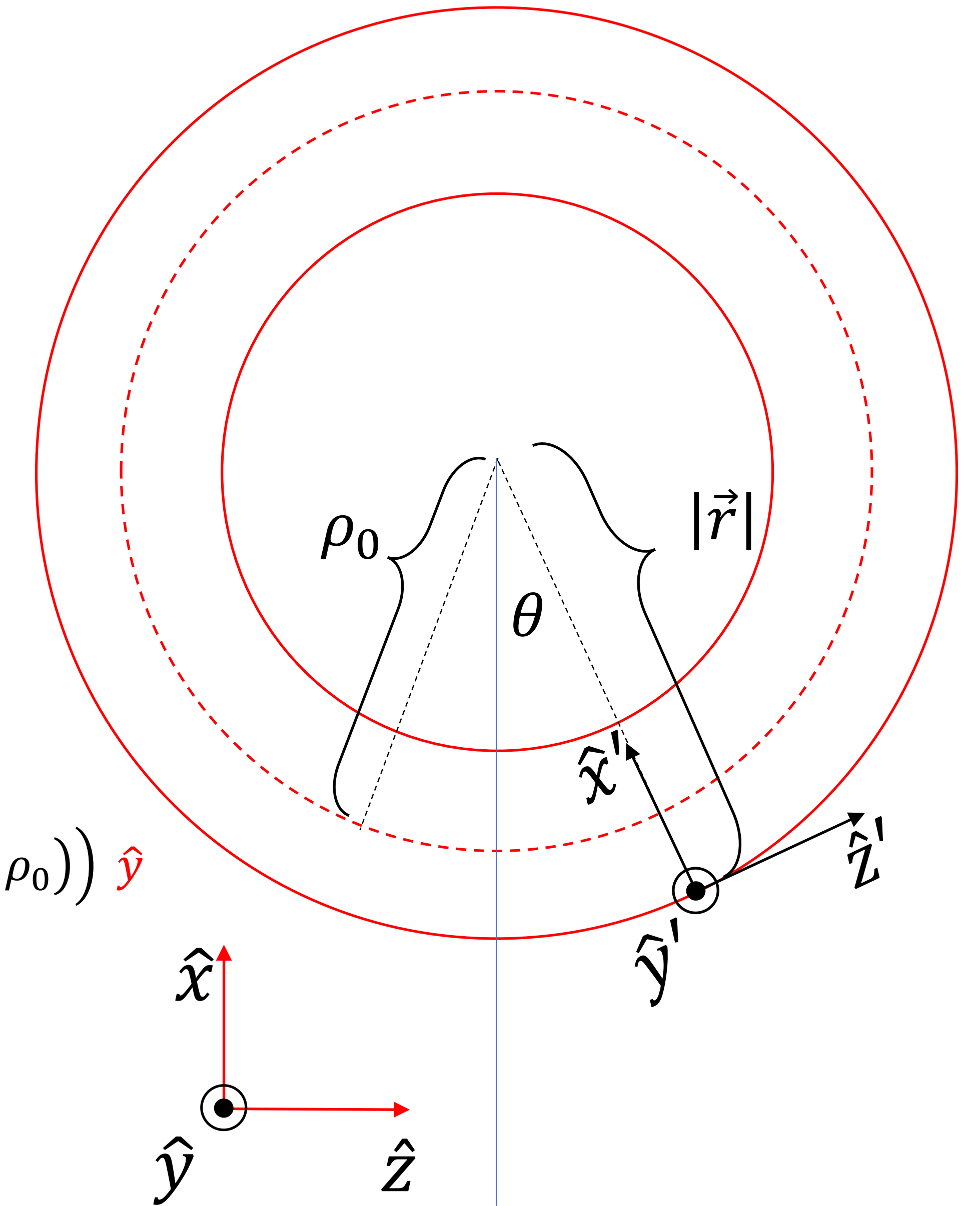
$$\vec{x}' = -(|\vec{r}| - \rho_0)\,\hat{x}' + y\,\hat{y}'$$

$$|\vec{r}| = \left|\vec{x} - \vec{O}_{curv}\right|$$

$$\hat{x}' = \mathbf{R}(\theta)\hat{x}, \quad \hat{z}' = \mathbf{R}(\theta)\hat{z}, \quad \hat{y}' = \hat{y}$$

$$\vec{B}'(\vec{x}') = (-2C_2 y)\,\mathbf{R}(\theta)\,\hat{x} + (C_1 - 2C_2 x')\,\hat{y}$$

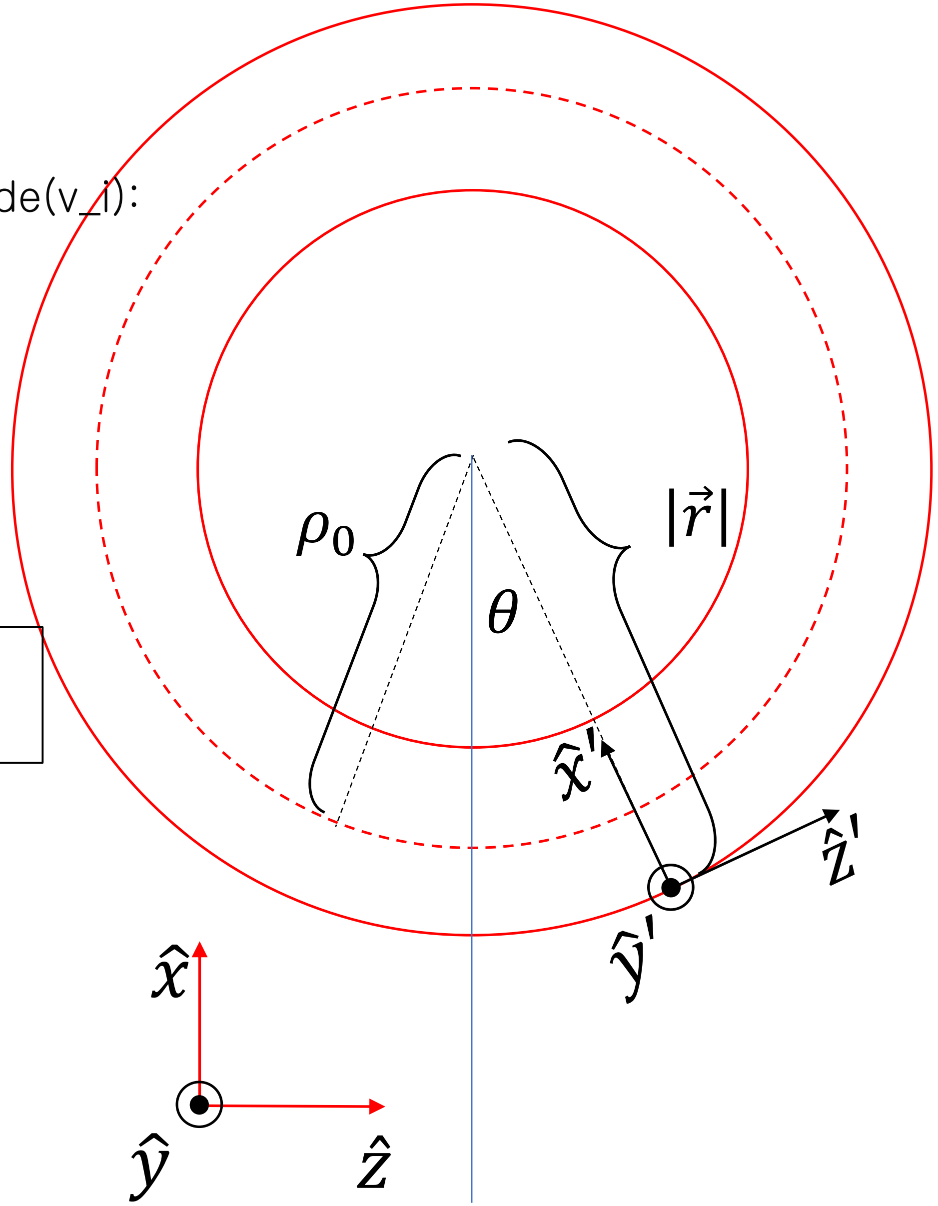
$$= (-2C_2 y)(\cos\theta\,\hat{x} - \sin\theta\,\hat{z}) + \Big(C_1 + 2C_2\big(\left|\vec{x} - \vec{O}_{curv}\right| - \rho_0\big)\Big)\,\hat{y}$$



2. Sector bend dipole + quadrupole (cont'd)

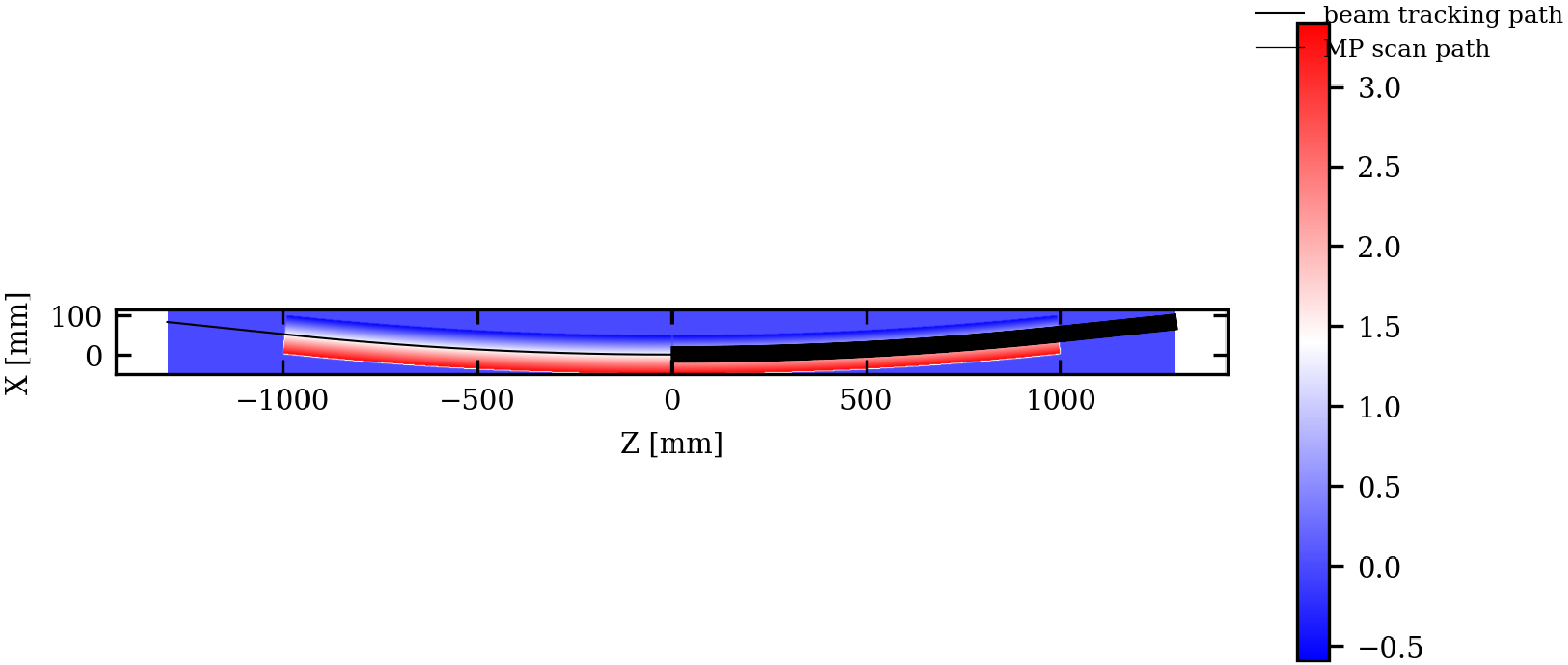
```
while x < xtab1:
    v_i = TwoVector(z, x)
    a = line_1.get_distance_to_point(v_i)
    b = line_2.get_distance_to_point(v_i)
    if a < 0 and b > 0 and not circ_ir.is_inside(v_i) and circ_or.is_inside(v_i):
        r = TwoVector(x,y) - v_curv_org
        theta = (-yhat2D).getTheta( r )

        shat_prime = zhat.rot_origin(yhat, theta)
        xhat_prime = xhat.rot_origin(yhat, theta) # equals to -rhat
        yhat_prime = yhat.new()

        x_prime = -(abs(r)-RHO)
        y_prime = y

        Bx_prime = -2.0 * C2 * y_prime
        By_prime = C1 -2.0 * C2 * x_prime
        B_prime = Bx_prime * xhat_prime + By_prime * yhat_prime

        Bx = B_prime.x
        By = B_prime.y
        Bz = B_prime.z
    else:
        Bx, By, Bz = 0, 0, 0
```

```
C1 = -1.4 * tesla
C2 = -20 * tesla/meter
```



2. Sector bend dipole + quadrupole (cont'd), r=20 mm

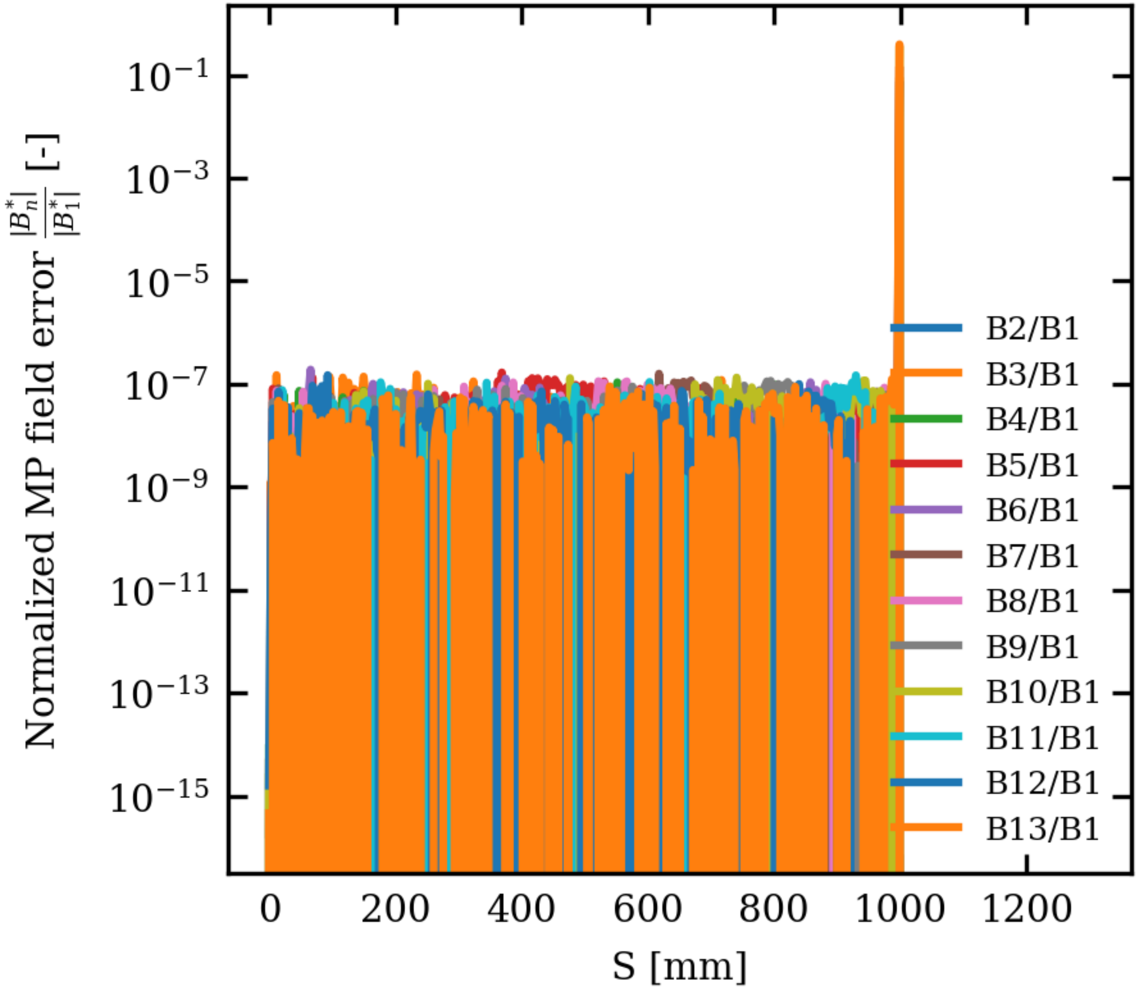
C1 = −1.4 * tesla
C2 = −20 * tesla/meter

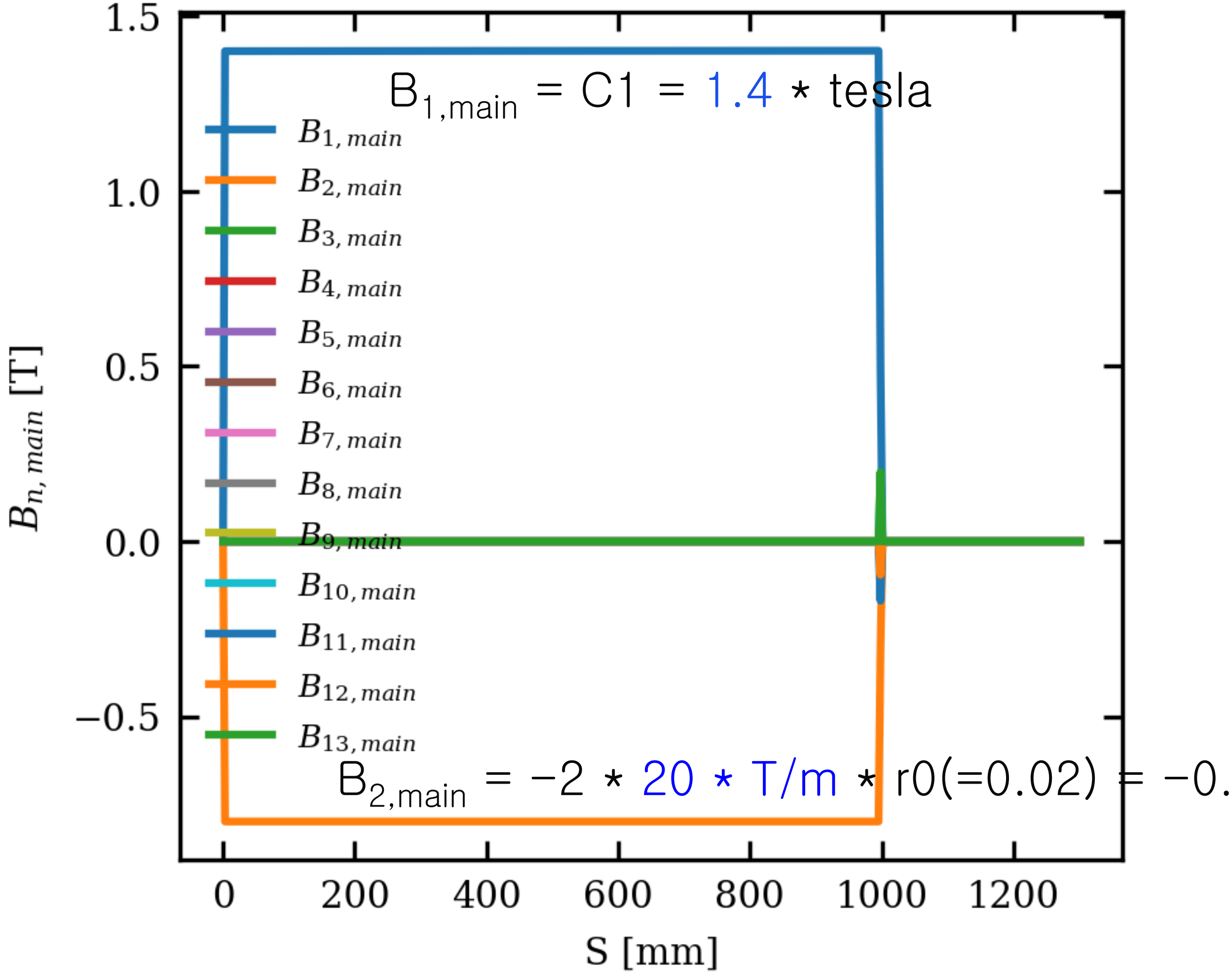


2. Sector bend dipole + quadrupole (cont'd), results, r=20 mm

$$\frac{|B_n^*|}{|B_N^*|}$$

$$B^*_{n,main}\Big|_{max} = \frac{1}{\pi}\oint\left(\vec{B}(\theta)\Big|_{r=r_0}\cdot\hat{r}\right)\sin n\theta\, d\theta$$

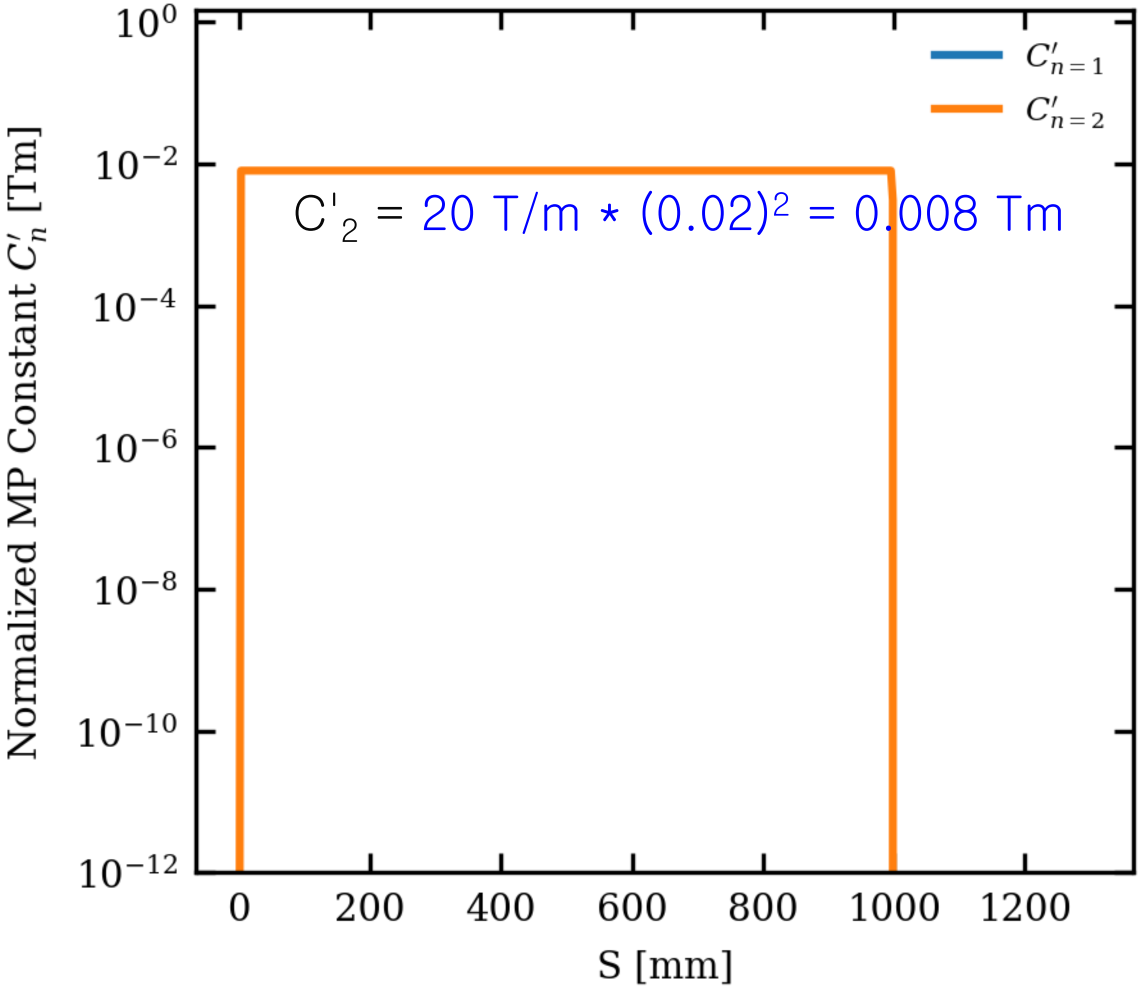


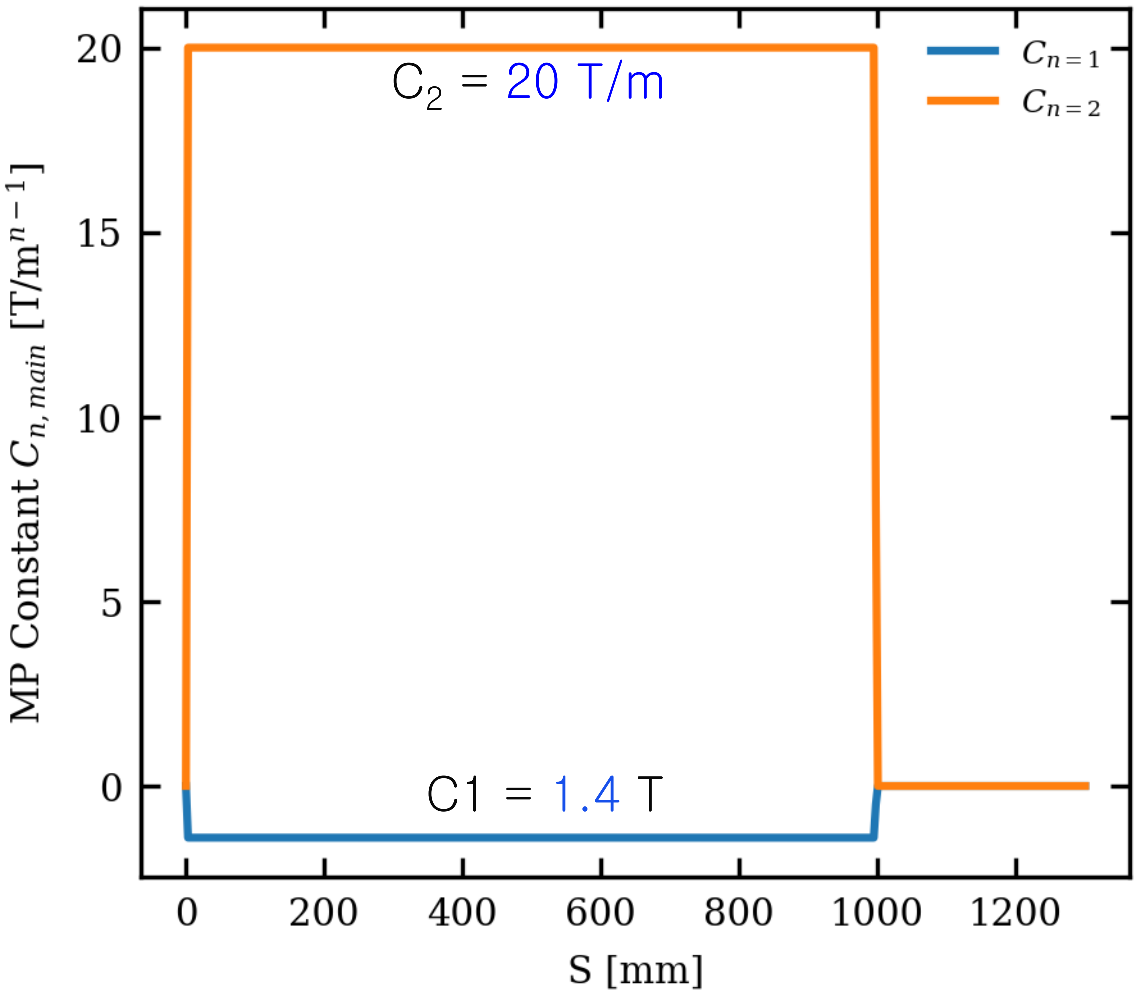


2. Sector bend dipole + quadrupole (cont'd), results , r=20 mm

$$C'_{n,main} = -\frac{r_0}{n} B^*_{n,main}\Big|_{max} \quad [\mathrm{T \cdot m}]$$

$$C_{n,main} = -\frac{1}{n r_0^{n-1}} B^*_{n,main}\Big|_{max} \quad [\mathrm{T/m^{n-1}}]$$

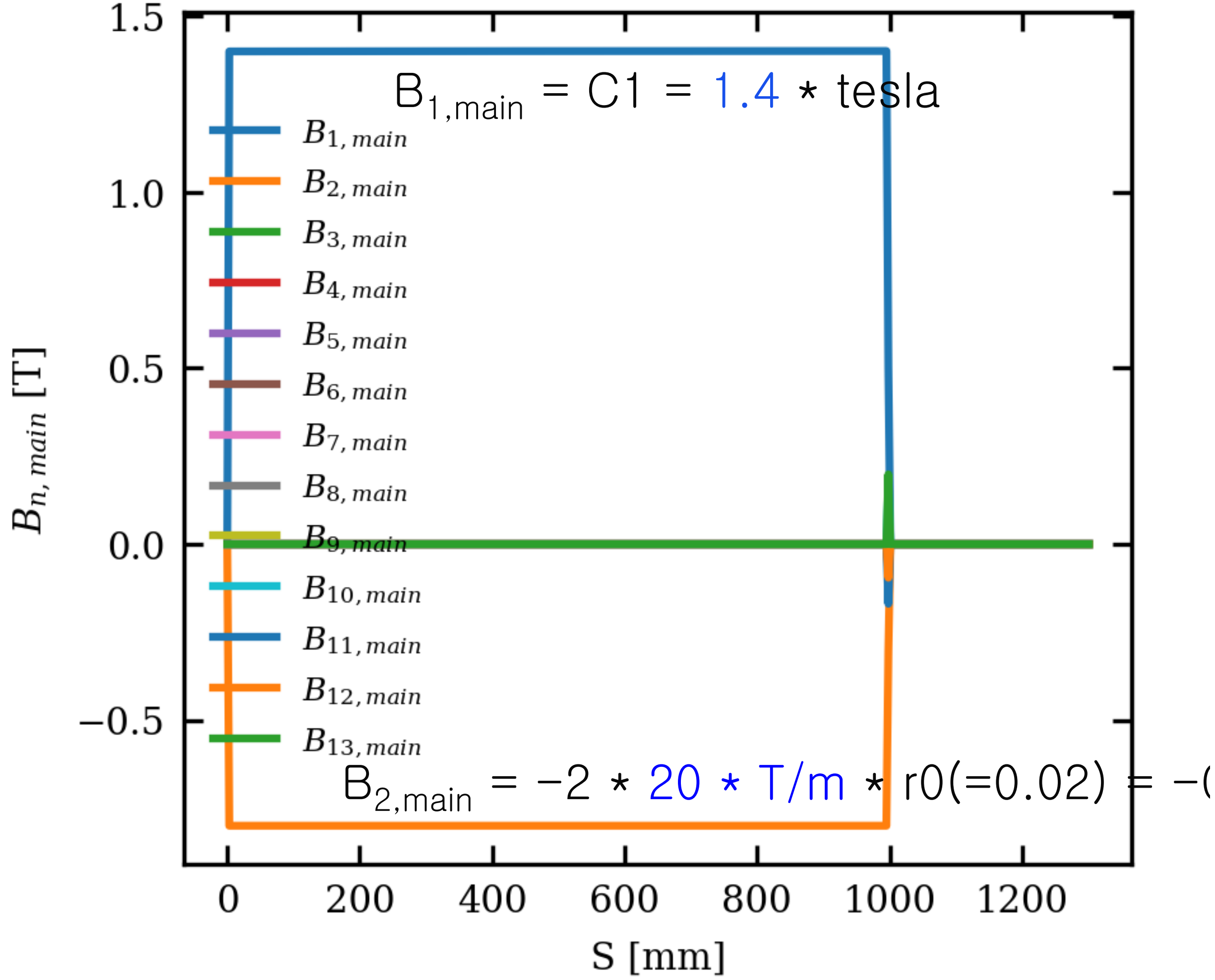


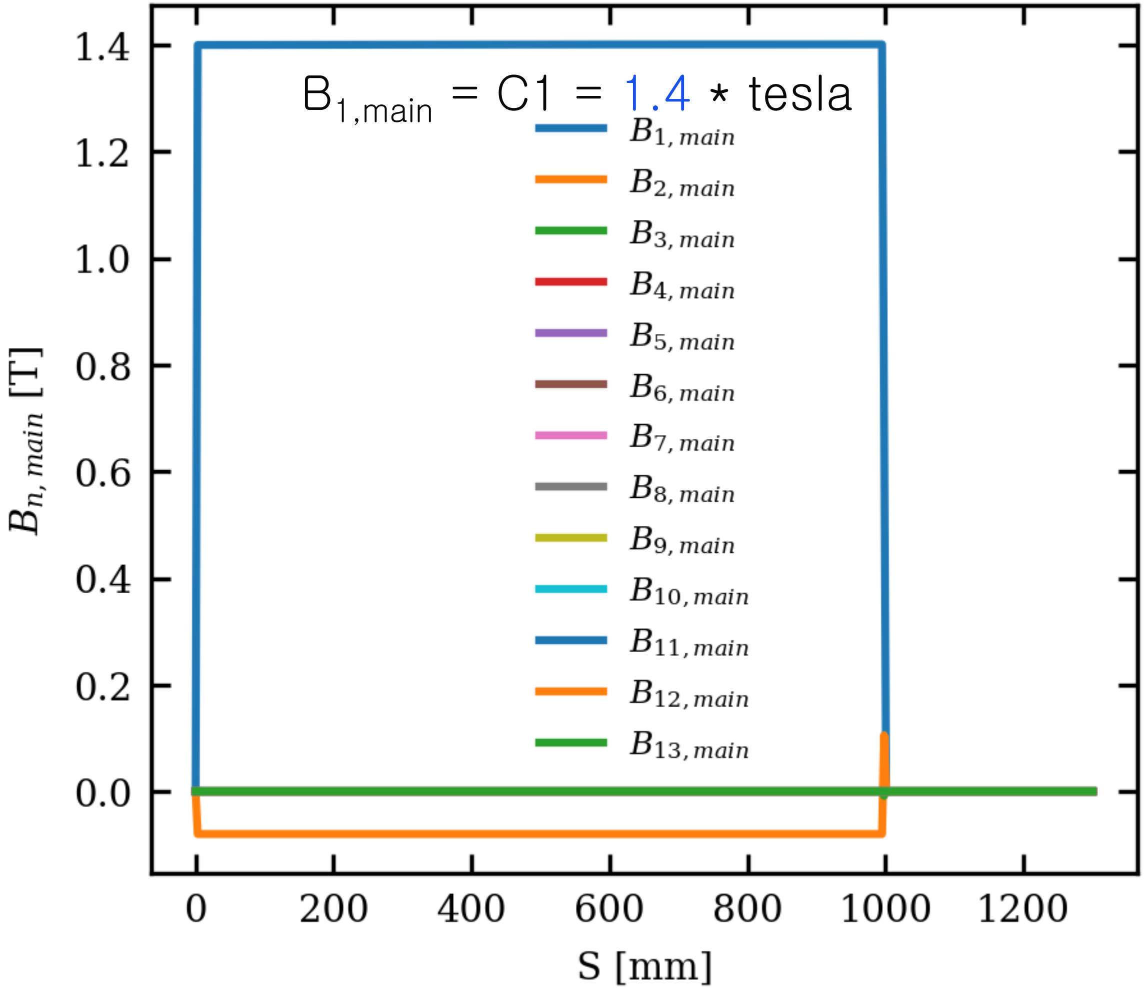


2. Sector bend dipole + quadrupole (cont'd), results , r=2 mm, 20 mm

$$\color{blue}{B^{*}_{n,main}\Big|_{max}} = \frac{1}{\pi}\oint\left(\vec{B}(\theta)\Big|_{r=r_0}\cdot\hat{r}\right)\sin n\theta\, d\theta$$

20 mm 경우가 멀티폴 신호 세기가 더 세니 오차가 확실히 작을 수 밖에 없음





B2,main = −2 * 20 * T/m * r0(=0.002) = −0.08