ANALYSIS AND MEASUREMENT OF DIPOLE AND QUADRUPOLE*

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Abstract

In accelerator systems, magnets play a critical role in guiding and focusing charged particles to maintain stable trajectories. To achieve this, various electromagnets such as dipoles, quadrupoles, and sextupoles are carefully designed, fabricated, and subjected to precise field measurements. Ensuring that the measured magnetic fields closely match the design specifications is essential for accelerator performance. This paper presents a basic analysis of magnet systems, focusing on dipole and quadrupole magnets, and describes the methods used for their field measurements.

INTRODUCTION

Magnets are essential components in accelerator systems, serving to steer and focus charged particles along stable orbits. To achieve precise control over particle trajectories, various types of electromagnets, such as dipoles, quadrupoles, and sextupoles, are employed. Each type of magnet plays a distinct role: dipoles provide bending forces, quadrupoles offer focusing forces, and sextupoles correct for chromatic aberrations.

The design, fabrication, and measurement of these magnets are critical processes to ensure that the actual magnetic fields closely match the theoretical requirements. Any deviation from the intended field quality can significantly impact the beam dynamics and the overall performance of the accelerator.

In this study, we focus on the basic analysis of dipole and quadrupole magnets. We describe the principles underlying their design and fabrication, and present field measurement techniques used to verify their performance against design specifications.

BASIC ANALYSIS OF MAGNETS

The role of magnets in an accelerator can be described fundamentally through the Lorentz force equation:

$$\vec{\mathrm{F}} = q(\vec{\mathrm{E}} + \vec{v} \times \vec{\mathrm{B}}), \tag{1}$$

where $q, \vec{v}$ is the charge and velocity of the particles, $\vec{\mathrm{E}}, \vec{\mathrm{B}}$ are the electric field and magnetic field, $\vec{\mathrm{F}}$ is the Lorentz force.

In most of the accelerator, magnets primarily exert a force that is perpendicular to the particle's direction of motion, effectively bending the trajectory of the charged particles. Before delving into detailed analysis, we briefly introduce the specific functions of different types of magnets used in accelerators.

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Magnets components

Magnets are used to bend the trajectory of charged particles. In regions where no magnetic field is present, the space is referred to as a **drift**. In a drift space, a particle moves freely according to its initial momentum without any external forces acting on it.

We define the transverse coordinates of a particle as x and y, and the longitudinal coordinate as z. Let s denote the path length along the reference trajectory, and x' represent the angle between the particle's trajectory and the s-axis in the x-plane.

The transverse motion of the particle in a drift space can be expressed as

$$x(s) = x_0 + x'_0 s, \tag{2}$$

where x_0 and x'_0 are the initial position and angle of the particle at the beginning of the drift.

A **dipole** magnet provides a uniform magnetic field that bends the trajectory of charged particles. Assuming that the magnetic field $\vec{B}$ is oriented along the vertical y direction, the Lorentz force causes the particle to bend in the horizontal x plane. This mechanism is shown in Fig. 1.

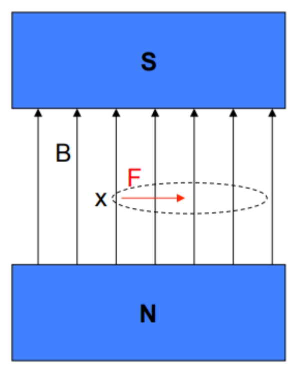

Figure 1: Basic structure of normal dipole magnet. The magnetic field strength is uniform across the magnet.

Neglecting the electric field, the equation of motion can be written as

$$x'' = \frac{1}{\rho}, \tag{3}$$

where ρ is the bending radius of the particle's trajectory determined by the magnetic field strength and particle momentum, shown as below:

$$B\rho = \frac{p}{q}, \tag{4}$$

where p is the particle momentum. The quantity $B\rho$, known as the **magnetic rigidity**, characterizes how resistant a charged particle is to bending under an external magnetic field.

A **quadrupole** magnet shown in Fig. 2 generates a magnetic field that varies linearly with the transverse position. Unlike dipoles, which bend the trajectory, quadrupoles focus or defocus the particle beam in transverse directions.

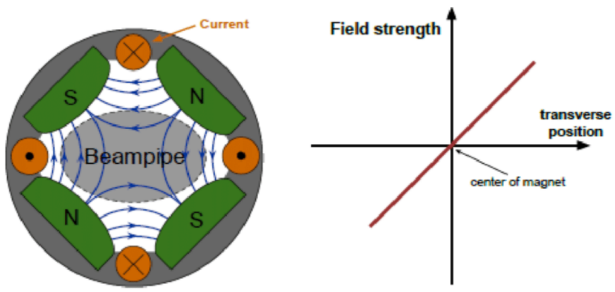

Figure 2: The basic structure of normal quadrupole magnet. The magnetic field strength increases linearly with the transverse position.

The ideal magnetic field components of a quadrupole are given by

$$B_x = Gy, \quad B_y = -Gx, \tag{5}$$

where G is the magnetic field gradient ($G = \frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y}$). The resulting Lorentz force leads to the following equations of motion [1]:

$$\frac{d^2x}{ds^2} + kx = 0, \tag{6}$$

$$\frac{d^2y}{ds^2} - ky = 0, \tag{7}$$

where the focusing strength k is defined as

$$k = \frac{qG}{p}. \tag{8}$$

In this configuration, the quadrupole acts as a focusing lens in one transverse plane and a defocusing lens in the other. However, as the transverse displacement of a particle increases, the focusing or defocusing force exerted by a quadrupole magnet also increases. By combining two quadrupole magnets with opposite focusing directions, net focusing in both transverse planes can be achieved.

In addition, **sextupole** magnets are used to correct chromatic aberrations, and **octupole** magnets are employed to control higher-order nonlinear effects. However, these components are beyond the scope of this paper and will not be discussed further.

Multipole analysis

In a magnetostatic with no current sources, the magnetic field satisfies the $\nabla \times B = 0$, and a magnetic scalar potential can be defined. In the complex plane, the potential can be decomposed into a real part $A(x, y)$ and an imaginary part $V(x, y)$, and represented as a complex potential F given by [2]

$$F(x, y) = A(x, y) + iV(x, y). \tag{9}$$

Since F acts as a scalar potential, its derivatives with respect to spatial coordinates yield the magnetic field components. The physical 2D coordinates are denoted as (x, y), and the position is represented using the complex variable.

$$z = x + iy. \tag{10}$$

The complex potential $F(z)$ is an analytic function of z, and the transverse magnetic field components can be obtained by differentiating F with respect to z.

The complex potential $F(z)$ can be expressed as a power series of z, where each power corresponds to a specific magnetic multipole order.

$$F(z) = \sum_{n=1}^{\infty} C_n z^n \tag{11}$$

In particular, $n = 1$ represents a dipole, $n = 2$ represents a quadrupole, etc. Using the polar representation of the complex variable z as

$$z = |z|e^{i\theta},$$

the form of $F(z)$ and the corresponding magnetic field B for each order n can be expressed as shown in Table 1.

Table 1: Multipole expansion of complex potential and corresponding magnetic field

Order n	Complex Potential $F(z)$	Magnetic Field B
1	$C_1\|z\|e^{i\theta}$	iC_1
2	$C_2\|z\|^2e^{i2\theta}$	$2iC_2\|z\|e^{i\theta}$
3	$C_3\|z\|^3e^{i3\theta}$	$3iC_3\|z\|^2e^{i2\theta}$

The function $F(z)$ satisfies the Cauchy-Riemann conditions, so this is an analytic function. As a result, the transformation from (x, y) to (A, V) can be interpreted as a conformal mapping. A conformal mapping preserves the angles between curves before and after the transformation.

In practical magnet design, magnets with higher-order multipoles such as quadrupoles, sextupoles, often exhibit complex geometrical structures. However, by employing conformal mappings, it is possible to simplify the design process: high-order magnetic field structures can be systematically generated from well-designed dipole fields through conformal mappings (see Fig. 3).

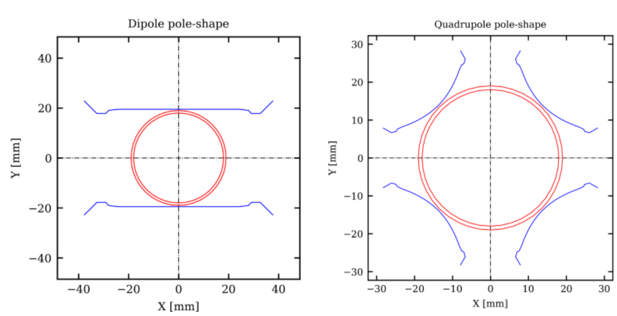

Figure 3: Transformation of a dipole-like structure into a quadrupole-like structure through the conformal mapping $w = z^2$.

MEASUREMENTS OF DIPOLE AND QUADRUPOLE MAGNETS

Once the magnet has been designed, it is essential to verify that the resulting magnetic field matches the design specifications. The magnetic field measurement methods are broadly classified into two categories: the rotating coil method and the Hall probe method. In the following sections, these techniques will be described in detail.

Rotating coil method

The rotating coil method is a widely used technique for measuring the magnetic field distribution, especially for extracting multipole components of magnets such as dipoles, quadrupoles, and sextupoles. In this method, a coil is placed inside the magnetic field and rotated mechanically around the center of the magnet.

As the coil rotates within the magnetic field, a voltage is induced according to Faraday's law of electromagnetic induction. The induced voltage is proportional to the rate of change of the magnetic flux through the coil. By recording the induced voltage as a function of the rotation angle, the magnetic flux can be reconstructed (see Fig. 4).

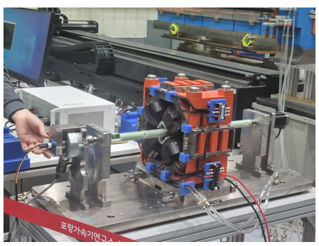

Figure 4: The rotating coil setup used in the experiment. A coil is placed inside the quadrupole magnet and rotated to measure the magnetic field.

The recorded signal can be decomposed using Fourier analysis, allowing the extraction of the multipole components present in the magnet. Each harmonic component in the Fourier series corresponds to a specific multipole: the first harmonic represents the dipole term, the second harmonic represents the quadrupole term, the third harmonic corresponds to the sextupole term, and so on.

This method provides high-precision measurements of integrated field quality and is particularly useful for characterizing the relative strengths and orientations of higher-order multipole errors.

Hall probe method

The Hall probe method is also used for measuring magnetic fields, particularly suitable for mapping the local magnetic field distribution. A Hall probe utilizes the Hall effect, where a voltage is generated across a conductor when it is placed in a magnetic field perpendicular to the current flow.

In this method, a Hall sensor is positioned at specific points within the magnet aperture to measure the local magnetic field components, typically B_x, B_y, and B_z. By moving the Hall probe along a path, a detailed field map can be obtained (see Fig. 5).

Figure 5: Hall probe system and the bending magnet used for magnetic field measurement. The Hall probe scans along the beam trajectory to measure the field profile.

This technique is particularly effective for capturing field variations near the magnet edges, fringe fields, and in regions where high spatial resolution is required. However, Hall probe measurements are generally more sensitive to noise compared to rotating coil measurements and may require careful calibration and correction procedures to achieve high accuracy.

The Hall probe method is mainly used for validating the field uniformity, detecting local field imperfections, and assisting in the alignment of magnet systems.

CONCLUSION

In this work, we introduced basic magnetic components such as drifts, dipoles, quadrupoles, and explained how their fields can be described using complex potentials and conformal mappings. After designing the magnets, it is important to check if the actual magnetic field matches the design. We briefly introduced two main measurement methods: the rotating coil method for extracting multipole components, and the Hall probe method for mapping local fields. These techniques help ensure that the magnets perform as expected in accelerator systems.

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